Vector Analysis

Vector analysis is a mathematical subject that is better taught by mathematicians than by engineers. Most junior and senior engineering students have not had the time (or the inclination) to take a course in vector analysis, although it is likely that vector concepts and operations were introduced in the calculus sequence. These are covered in this chapter, and the time devoted to them now should depend on past exposure.

1.1 SCALARS AND VECTORS

The term *scalar* refers to a quantity whose value may be represented by a single (positive or negative) real number. The x, y, and z we use in basic algebra are scalars, and the quantities they represent are scalars. If we speak of a body falling a distance L in a time t, or the temperature T at any point in a bowl of soup whose coordinates are x, y, and z, then L, t, T, x, y, and z are all scalars. Other scalar quantities are mass, density, pressure (but not force), volume, volume resistivity, and voltage.

1.2 VECTOR ALGEBRA

With the definition of vectors and vector fields now established, we may proceed to define the rules of vector arithmetic, vector algebra, and (later) vector calculus. Some of the rules will be similar to those of scalar algebra, some will differ slightly, and some will be entirely new.

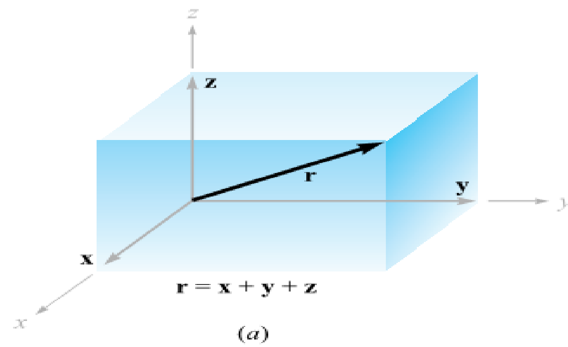
To begin, the addition of vectors follows the parallelogram law. Figure 1.1 shows the sum of two vectors, **A** and **B**. It is easily seen that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$, or that vector addition obeys the commutative law. Vector addition also obeys the associative law,

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

1.4 VECTOR COMPONENTS AND UNIT VECTORS

If the component vectors of the vector **r** are **x**, **y**, and **z**,

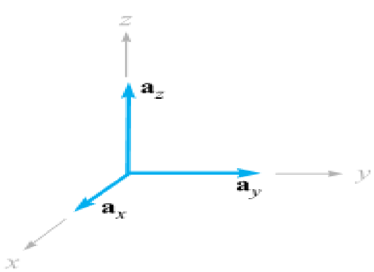
then $\mathbf{r} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. The component vectors are shown in Figure 1.3*a*. Instead of one vector, we now have three, but this is a step forward because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes.



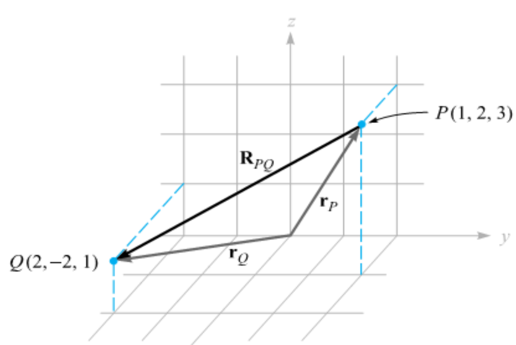
(a)

The component vectors have magnitudes that depend on the given vector (such as **r**), but they each have a known and constant direction. This suggests the use of *unit vectors* having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values. We reserve the symbol **a** for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system.[3] They are directed along the x, y, and z axes, respectively, as shown in Figure 1.3*b*.

If the component vector **y** happens to be two units in magnitude and directed toward increasing values of y, we should then write $\mathbf{y} = 2\mathbf{a}_y$. A vector $\mathbf{r}_P$ pointing



(b)



(c)

from the origin to point $P(1, 2, 3)$ is written $\mathbf{r}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$. The vector from P to Q may be obtained by applying the rule of vector addition. This rule shows that the vector from the origin to P plus the vector from P to Q is equal to the vector from the origin to Q. The desired vector from $P(1, 2, 3)$ to $Q(2, -2, 1)$ is therefore

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$
$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

The vectors $\mathbf{r}_P$, $\mathbf{r}_Q$, and $\mathbf{R}_{PQ}$ are shown in Figure 1.3*c*.

Any vector $\mathbf{B}$ then may be described by $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The magnitude of $\mathbf{B}$ written $|\mathbf{B}|$ or simply B, is given by

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} \tag{1}$$

Each of the three coordinate systems we discuss will have its three fundamental and mutually perpendicular unit vectors that are used to resolve any vector into its component vectors. Unit vectors are not limited to this application. It is helpful to write a unit vector having a specified direction. This is easily done, for a unit vector in a given direction is merely a vector in that direction divided by its magnitude. A unit vector in the $\mathbf{r}$ direction is $\mathbf{r}/\sqrt{x^2 + y^2 + z^2}$, and a unit vector in the direction of the vector $\mathbf{B}$ is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|} \tag{2}$$

D1.1. Given points $M(-1, 2, 1)$, $N(3, -3, 0)$, and $P(-2, -3, -4)$, find: (*a*) $\mathbf{R}_{MN}$; (*b*) $\mathbf{R}_{MN} + \mathbf{R}_{MP}$; (*c*) $|\mathbf{r}_M|$; (*d*) $\mathbf{a}_{MP}$; (*e*) $|2\mathbf{r}_P - 3\mathbf{r}_N|$.

Ans. $4\mathbf{a}_x - 5\mathbf{a}_y - \mathbf{a}_z$; $3\mathbf{a}_x - 10\mathbf{a}_y - 6\mathbf{a}_z$; 2.45; $-0.14\mathbf{a}_x - 0.7\mathbf{a}_y - 0.7\mathbf{a}_z$; 15.56

EXAMPLE 1.1

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$.

Solution. We first construct the vector extending from the origin to point G,

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

We continue by finding the magnitude of **G**,

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

and finally expressing the desired unit vector as the quotient,

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

A special symbol is desirable for a unit vector so that its character is immediately apparent. Symbols that have been used are $\mathbf{u}_B$, $\mathbf{a}_B$, $\mathbf{1}_B$, or even **b**. We will consistently use the lowercase **a** with an appropriate subscript.

THE DOT PRODUCT

We now consider the first of two types of vector multiplication. The second type will be discussed in the following section.

Given two vectors **A** and **B**, the *dot product*, or *scalar product*, is defined as the product of the magnitude of **A**, the magnitude of **B**, and the cosine of the smaller angle between them,

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \cos\theta_{AB} \tag{3}$$

The dot appears between the two vectors and should be made heavy for emphasis. The dot, or scalar, product is a scalar, as one of the names implies, and it obeys the commutative law,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{4}$$

Finding the angle between two vectors in three-dimensional space is often a job we would prefer to avoid, and for that reason the definition of the dot product is usually not used in its basic form. A more helpful result is obtained by considering two vectors whose rectangular components are given, such as $\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$ and $\mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z$. The dot product also obeys the distributive law, and, therefore, $\mathbf{A} \cdot \mathbf{B}$ yields the sum of nine scalar terms, each involving the dot product of two unit vectors. Because the angle between two different unit vectors of the rectangular coordinate system is 90°, we then have

$$\mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_x = \mathbf{a}_x \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_z \cdot \mathbf{a}_y = 0$$

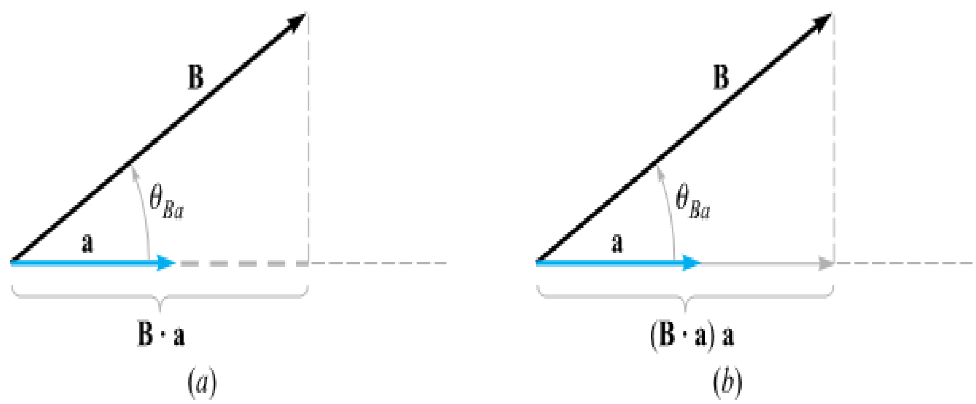


Figure 1.4 (*a*) The scalar component of **B** in the direction of the unit vector **a** is **B** · **a**. (*b*) The vector component of **B** in the direction of the unit vector **a** is (**B** · **a**)**a**.

The remaining three terms involve the dot product of a unit vector with itself, which is unity, giving finally

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{5}$$

which is an expression involving no angles.

A vector dotted with itself yields the magnitude squared, or

$$\mathbf{A} \cdot \mathbf{A} = A^2 = |\mathbf{A}|^2 \tag{6}$$

and any unit vector dotted with itself is unity,

$$\mathbf{a}_A \cdot \mathbf{a}_A = 1$$

One of the most important applications of the dot product is that of finding the component of a vector in a given direction. Referring to Figure 1.4*a*, we can obtain the component (scalar) of **B** in the direction specified by the unit vector **a** as

$$\mathbf{B} \cdot \mathbf{a} = |\mathbf{B}|\,|\mathbf{a}| \cos\theta_{Ba} = |\mathbf{B}| \cos\theta_{Ba}$$

The sign of the component is positive if $0 \leq \theta_{Ba} \leq 90°$ and negative whenever $90° \leq \theta_{Ba} \leq 180°$.

To obtain the component *vector* of **B** in the direction of **a**, we multiply the component (scalar) by **a**, as illustrated by Figure 1.4*b*. For example, the component of **B** in the direction of $\mathbf{a}_x$ is $\mathbf{B} \cdot \mathbf{a}_x = B_x$, and the component vector is $B_x \mathbf{a}_x$, or $(\mathbf{B} \cdot \mathbf{a}_x)\mathbf{a}_x$. Hence, the problem of finding the component of a vector in any direction becomes the problem of finding a unit vector in that direction, and that we can do.

The geometrical term *projection* is also used with the dot product. Thus, $\mathbf{B} \cdot \mathbf{a}$ is the projection of **B** in the **a** direction.

EXAMPLE 1.2

In order to illustrate these definitions and operations, consider the vector field $\mathbf{G} = y\mathbf{a}_x - 2.5x\mathbf{a}_y + 3\mathbf{a}_z$ and the point $Q(4, 5, 2)$. We wish to find: $\mathbf{G}$ at Q; the scalar component of $\mathbf{G}$ at Q in the direction of $\mathbf{a}_N = \frac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z)$; the vector component of $\mathbf{G}$ at Q in the direction of $\mathbf{a}_N$; and finally, the angle θ_{Ga} between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$.

Solution. Substituting the coordinates of point Q into the expression for $\mathbf{G}$, we have

$$\mathbf{G}(\mathbf{r}_Q) = 5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z$$

Next we find the scalar component. Using the dot product, we have

$$\mathbf{G} \cdot \mathbf{a}_N = (5\mathbf{a}_x - 10\mathbf{a}_y + 3\mathbf{a}_z) \cdot \tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = \tfrac{1}{3}(10 - 10 - 6) = -2$$

The vector component is obtained by multiplying the scalar component by the unit vector in the direction of $\mathbf{a}_N$,

$$(\mathbf{G} \cdot \mathbf{a}_N)\mathbf{a}_N = -(2)\tfrac{1}{3}(2\mathbf{a}_x + \mathbf{a}_y - 2\mathbf{a}_z) = -1.333\mathbf{a}_x - 0.667\mathbf{a}_y + 1.333\mathbf{a}_z$$

The angle between $\mathbf{G}(\mathbf{r}_Q)$ and $\mathbf{a}_N$ is found from

$$\mathbf{G} \cdot \mathbf{a}_N = |\mathbf{G}| \cos\theta_{Ga}$$
$$-2 = \sqrt{25 + 100 + 9}\cos\theta_{Ga}$$

and

$$\theta_{Ga} = \cos^{-1}\frac{-2}{\sqrt{134}} = 99.9^\circ$$

D1.3. The three vertices of a triangle are located at $A(6, -1, 2)$, $B(-2, 3, -4)$, and $C(-3, 1, 5)$. Find: (*a*) $\mathbf{R}_{AB}$; (*b*) $\mathbf{R}_{AC}$; (*c*) the angle θ_{BAC} at vertex A; (*d*) the (vector) projection of $\mathbf{R}_{AB}$ on $\mathbf{R}_{AC}$.

Ans. $-8\mathbf{a}_x + 4\mathbf{a}_y - 6\mathbf{a}_z$; $-9\mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$; 53.6°; $-5.94\mathbf{a}_x + 1.319\mathbf{a}_y + 1.979\mathbf{a}_z$

THE CROSS PRODUCT

Given two vectors **A** and **B**, we now define the *cross product*, or *vector product*, of **A** and **B**, written with a cross between the two vectors as $\mathbf{A} \times \mathbf{B}$ and read "**A** cross **B**." The cross product $\mathbf{A} \times \mathbf{B}$ is a vector; the magnitude of $\mathbf{A} \times \mathbf{B}$ is equal to the product of the magnitudes of **A**, **B**, and the sine of the smaller angle between **A** and **B**; the direction of $\mathbf{A} \times \mathbf{B}$ is perpendicular to the plane containing **A** and **B** and is along one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as **A** is turned into **B**. This direction is illustrated in Figure 1.5. Remember that either vector may be moved about at will, maintaining its direction constant, until the two vectors have a "common origin." This determines the plane containing both. However, in most of our applications we will be concerned with vectors defined at the same point.

As an equation we can write

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB} \tag{7}$$

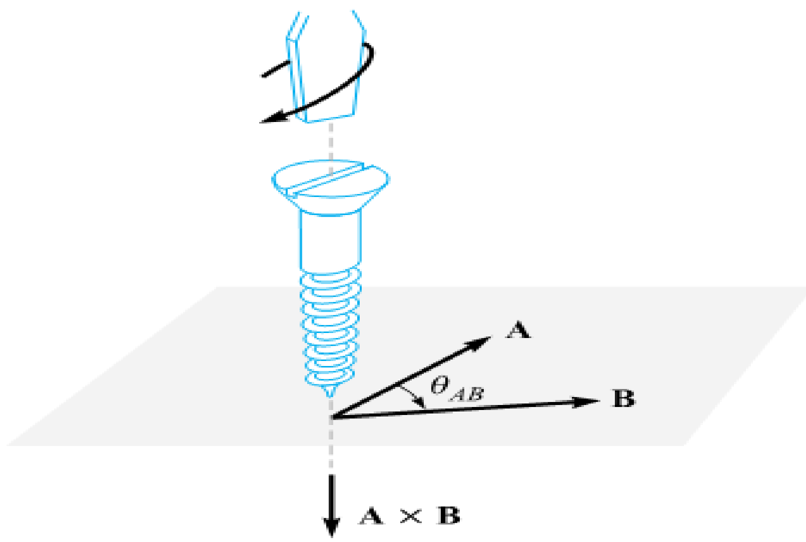


Figure 1.5 The direction of A × B is in the direction of advance of a right-handed screw as A is turned into B.

$$\mathbf{A}\times\mathbf{B} = A_xB_x\mathbf{a}_x\times\mathbf{a}_x + A_xB_y\mathbf{a}_x\times\mathbf{a}_y + A_xB_z\mathbf{a}_x\times\mathbf{a}_z$$
$$+ A_yB_x\mathbf{a}_y\times\mathbf{a}_x + A_yB_y\mathbf{a}_y\times\mathbf{a}_y + A_yB_z\mathbf{a}_y\times\mathbf{a}_z$$
$$+ A_zB_x\mathbf{a}_z\times\mathbf{a}_x + A_zB_y\mathbf{a}_z\times\mathbf{a}_y + A_zB_z\mathbf{a}_z\times\mathbf{a}_z$$

We have already found that $\mathbf{a}_x\times\mathbf{a}_y = \mathbf{a}_z$, $\mathbf{a}_y\times\mathbf{a}_z = \mathbf{a}_x$, and $\mathbf{a}_z\times\mathbf{a}_x = \mathbf{a}_y$. The three remaining terms are zero, for the cross product of any vector with itself is zero, since the included angle is zero. These results may be combined to give

$$\mathbf{A}\times\mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z \tag{8}$$

or written as a determinant in a more easily remembered form,

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{9}$$

Thus, if $\mathbf{A} = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ and $\mathbf{B} = -4\mathbf{a}_x - 2\mathbf{a}_y + 5\mathbf{a}_z$, we have

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 2 & -3 & 1 \\ -4 & -2 & 5 \end{vmatrix}$$

$$= [(-3)(5) - (1(-2)]\mathbf{a}_x - [(2)(5) - (1)(-4)]\mathbf{a}_y + [(2)(-2) - (-3)(-4)]\mathbf{a}_z$$

$$= -13\mathbf{a}_x - 14\mathbf{a}_y - 16\mathbf{a}_z$$

D1.4. The three vertices of a triangle are located at $A(6, -1, 2)$, $B(-2, 3, -4)$, and $C(-3, 1, 5)$. Find: (*a*) $\mathbf{R}_{AB}\times\mathbf{R}_{AC}$; (*b*) the area of the triangle; (*c*) a unit vector perpendicular to the plane in which the triangle is located.

Ans. $24\mathbf{a}_x + 78\mathbf{a}_y + 20\mathbf{a}_z$; 42.0; $0.286\mathbf{a}_x + 0.928\mathbf{a}_y + 0.238\mathbf{a}_z$

OTHER COORDINATE SYSTEMS: CIRCULAR CYLINDRICAL COORDINATES

The variables of the rectangular and cylindrical coordinate systems are easily related to each other. Referring to Figure 1.7, we see that

$$\begin{aligned} x &= \rho \cos\phi \\ y &= \rho \sin\phi \\ z &= z \end{aligned} \tag{10}$$

From the other viewpoint, we may express the cylindrical variables in terms of x, y, and z:

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \quad (\rho \geq 0) \\ \phi &= \tan^{-1}\frac{y}{x} \\ z &= z \end{aligned} \tag{11}$$

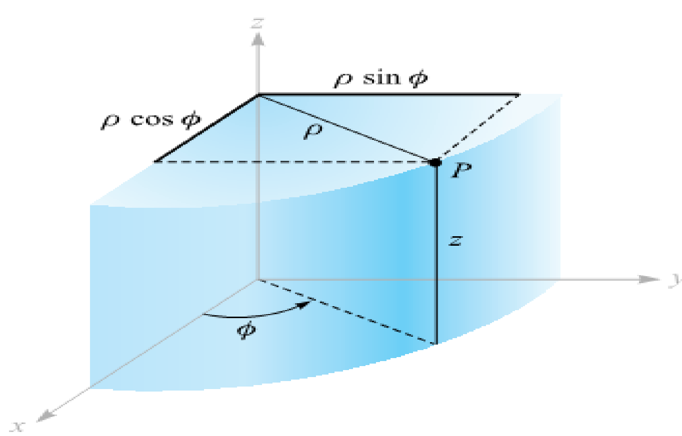


Figure 1.7 The relationship between the rectangular variables x, y, z and the cylindrical coordinate variables ρ, ϕ, z. There is no change in the variable z between the two systems.

vectors is generally required. That is, we may be given a rectangular vector

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

where each component is given as a function of x, y, and z, and we need a vector in cylindrical coordinates

$$\mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

where each component is given as a function of ρ, ϕ, and z.

To find any desired component of a vector, we recall from the discussion of the dot product that a component in a desired direction may be obtained by taking the dot product of the vector and a unit vector in the desired direction. Hence,

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho \qquad \text{and} \qquad A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi$$

Expanding these dot products, we have

$$A_\rho = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\rho = A_x\mathbf{a}_x\cdot\mathbf{a}_\rho + A_y\mathbf{a}_y\cdot\mathbf{a}_\rho \tag{12}$$

$$A_\phi = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_\phi = A_x\mathbf{a}_x\cdot\mathbf{a}_\phi + A_y\mathbf{a}_y\cdot\mathbf{a}_\phi \tag{13}$$

and

$$A_z = (A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z)\cdot\mathbf{a}_z = A_z\mathbf{a}_z\cdot\mathbf{a}_z = A_z \tag{14}$$

since $\mathbf{a}_z\cdot\mathbf{a}_\rho$ and $\mathbf{a}_z\cdot\mathbf{a}_\phi$ are zero.

Table 1.1 Dot products of unit vectors in cylindrical and rectangular coordinate systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x\cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y\cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z\cdot$	0	0	1

EXAMPLE 1.3

Transform the vector $\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$ into cylindrical coordinates.

Solution. The new components are

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$
$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$
$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$
$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

Thus,

$$\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z$$

D1.5. (*a*) Give the rectangular coordinates of the point $C(\rho = 4.4, \phi = -115°, z = 2)$. (*b*) Give the cylindrical coordinates of the point $D(x = -3.1, y = 2.6, z = -3)$. (*c*) Specify the distance from C to D.

Ans. $C(x = -1.860, y = -3.99, z = 2)$; $D(\rho = 4.05, \phi = 140.0°, z = -3)$; 8.36

D1.6. Transform to cylindrical coordinates: (*a*) $\mathbf{F} = 10\mathbf{a}_x - 8\mathbf{a}_y + 6\mathbf{a}_z$ at point $P(10, -8, 6)$; (*b*) $\mathbf{G} = (2x + y)\mathbf{a}_x - (y - 4x)\mathbf{a}_y$ at point $Q(\rho, \phi, z)$. (*c*) Give the rectangular components of the vector $\mathbf{H} = 20\mathbf{a}_\rho - 10\mathbf{a}_\phi + 3\mathbf{a}_z$ at $P(x = 5, y = 2, z = -1)$.

Ans. $12.81\mathbf{a}_\rho + 6\mathbf{a}_z$; $(2\rho\cos^2\phi - \rho\sin^2\phi + 5\rho\sin\phi\cos\phi)\mathbf{a}_\rho + (4\rho\cos^2\phi - \rho\sin^2\phi - 3\rho\sin\phi\cos\phi)\mathbf{a}_\phi$; $H_x = 22.3$, $H_y = -1.857$, $H_z = 3$

THE SPHERICAL COORDINATE SYSTEM

The transformation of scalars from the rectangular to the spherical coordinate system is easily made by using Figure 1.8*a* to relate the two sets of variables:

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi \qquad (15)$$
$$z = r\cos\theta$$

The transformation in the reverse direction is achieved with the help of

$$r = \sqrt{x^2+y^2+z^2} \qquad (r \geq 0)$$
$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}} \qquad (0° \leq \theta \leq 180°) \qquad (16)$$
$$\phi = \tan^{-1}\frac{y}{x}$$

The radius variable r is nonnegative, and θ is restricted to the range from 0° to 180°, inclusive. The angles are placed in the proper quadrants by inspecting the signs of x, y, and z.

Table 1.2 Dot products of unit vectors in spherical and rectangular coordinate systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x\cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y\cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z\cdot$	$\cos\theta$	$-\sin\theta$	0

vector in the direction of the rectangular vector, the dot products with $\mathbf{a}_z$ are found to be

$$\mathbf{a}_z\cdot\mathbf{a}_r = \cos\theta$$
$$\mathbf{a}_z\cdot\mathbf{a}_\theta = -\sin\theta$$
$$\mathbf{a}_z\cdot\mathbf{a}_\phi = 0$$

The dot products involving $\mathbf{a}_x$ and $\mathbf{a}_y$ require first the projection of the spherical unit vector on the xy plane and then the projection onto the desired axis. For example, $\mathbf{a}_r\cdot\mathbf{a}_x$ is obtained by projecting $\mathbf{a}_r$ onto the xy plane, giving $\sin\theta$, and then projecting $\sin\theta$ on the x axis, which yields $\sin\theta\cos\phi$. The other dot products are found in a like manner, and all are shown in Table 1.2.

EXAMPLE 1.4

We illustrate this procedure by transforming the vector field $\mathbf{G} = (xz/y)\mathbf{a}_x$ into spherical components and variables.

Solution. We find the three spherical components by dotting $\mathbf{G}$ with the appropriate unit vectors, and we change variables during the procedure:

$$G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y} \sin\theta \cos\phi$$

$$= r \sin\theta \cos\theta \frac{\cos^2\phi}{\sin\phi}$$

$$G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y} \cos\theta \cos\phi$$

$$= r \cos^2\theta \frac{\cos^2\phi}{\sin\phi}$$

$$G\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi)$$

$$= -r \cos\theta \cos\phi$$

Collecting these results, we have

$$\mathbf{G} = r \cos\theta \cos\phi \,(\sin\theta \cot\phi \,\mathbf{a}_r + \cos\theta \cot\phi \,\mathbf{a}_\theta - \mathbf{a}_\phi)$$

D1.7. Given the two points, $C(-3, 2, 1)$ and $D(r = 5, \theta = 20°, \phi = -70°)$, find: (*a*) the spherical coordinates of C; (*b*) the rectangular coordinates of D; (*c*) the distance from C to D.

Ans. $C(r = 3.74, \theta = 74.5°, \phi = 146.3°)$; $D(x = 0.585, y = -1.607, z = 4.70)$; 6.29

D1.8. Transform the following vectors to spherical coordinates at the points given: (*a*) $10\mathbf{a}_x$ at $P(x = -3, y = 2, z = 4)$; (*b*) $10\mathbf{a}_y$ at $Q(\rho = 5, \phi = 30°, z = 4)$; (*c*) $10\mathbf{a}_z$ at $M(r = 4, \theta = 110°, \phi = 120°)$.

Ans. $-5.57\mathbf{a}_r - 6.18\mathbf{a}_\theta - 5.55\mathbf{a}_\phi$; $3.90\mathbf{a}_r + 3.12\mathbf{a}_\theta + 8.66\mathbf{a}_\phi$; $-3.42\mathbf{a}_r - 9.40\mathbf{a}_\theta$

A.3 VECTOR IDENTITIES

The vector identities that follow may be proved by expansion in rectangular (or general curvilinear) coordinates. The first two identities involve the scalar and vector triple products, the next three are concerned with operations on sums, the following three apply to operations when the argument is multiplied by a scalar function, the next three apply to operations on scalar or vector products, and the last four concern the second-order operations.

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \equiv (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} \equiv (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B} \tag{A.6}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \equiv (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{A.7}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) \equiv \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{A.8}$$

$$\nabla(V + W) \equiv \nabla V + \nabla W \tag{A.9}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) \equiv \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{A.10}$$

$$\nabla \cdot (V\mathbf{A}) \equiv \mathbf{A} \cdot \nabla V + V \nabla \cdot \mathbf{A} \tag{A.11}$$

$$\nabla(VW) \equiv V\nabla W + W\nabla V \tag{A.12}$$

$$\nabla \times (V\mathbf{A}) \equiv \nabla V \times \mathbf{A} + V\nabla \times \mathbf{A} \tag{A.13}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{A.14}$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) \equiv (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \tag{A.15}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{A.16}$$

$$\nabla \cdot \nabla V \equiv \nabla^2 V \tag{A.17}$$

$$\nabla \cdot \nabla \times \mathbf{A} \equiv 0 \tag{A.18}$$

$$\nabla \times \nabla V \equiv 0 \tag{A.19}$$

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{A.20}$$

Units

The force between the two parallel conductors is known to be

$$F = \mu_0 \frac{I^2}{2\pi d}$$

and thus

$$2 \times 10^{-7} = \mu_0 \frac{1}{2\pi}$$

or

$$\mu_0 = 4\pi \times 10^{-7} \qquad (\text{kg} \cdot \text{m/A}^2 \cdot \text{s}^2, \text{ or H/m})$$

with which an electromagnetic wave propagates in free space is

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

and thus

$$\epsilon_0 = \frac{1}{\mu_0 c^2} = \frac{1}{4\pi 10^{-7} c^2} = 8.854\,187\,817 \times 10^{-12} \text{ F/m}$$

Table B.1 Names and units of the electric and magnetic quantities in the International System (in the order in which they appear in the text)

Symbol	Name	Unit	Abbreviation
v	Velocity	meter/second	m/s
F	Force	newton	N
Q	Charge	coulomb	C
r, R	Distance	meter	m
ϵ_0, ϵ	Permittivity	farad/meter	F/m
E	Electric field intensity	volt/meter	V/m
ρ_v	Volume charge density	coulomb/meter^3	C/m^3
v	Volume	meter^3	m^3
ρ_L	Linear charge density	coulomb/meter	C/m
ρ_S	Surface charge density	coulomb/meter^2	C/m^2
Ψ	Electric flux	coulomb	C
D	Electric flux density	coulomb/meter^2	C/m^2
S	Area	meter^2	m^2
W	Work, energy	joule	J
L	Length	meter	m
V	Potential	volt	V
p	Dipole moment	coulomb-meter	C·m
I	Current	ampere	A
J	Current density	ampere/meter^2	A/m^2
μ_e, μ_h	Mobility	meter^2/volt-second	m^2/V·s
e	Electronic charge	coulomb	C
σ	Conductivity	siemens/meter	S/m
R	Resistance	ohm	Ω
P	Polarization	coulomb/meter^2	C/m^2
$\chi_{e,m}$	Susceptibility		
C	Capacitance	farad	F
R_s	Sheet resistance	ohm per square	Ω
H	Magnetic field intensity	ampere/meter	A/m
K	Surface current density	ampere/meter	A/m
B	Magnetic flux density	tesla (or weber/meter^2)	T (or Wb/m^2)
μ_0, μ	Permeability	henry/meter	H/m
Φ	Magnetic flux	weber	Wb
V_m	Magnetic scalar potential	ampere	A
A	Vector magnetic potential	weber/meter	Wb/m
T	Torque	newton-meter	N·m
m	Magnetic moment	ampere-meter^2	A·m^2

Table B.3 Standard prefixes used with SI units

Prefix	Abbrev.	Meaning	Prefix	Abbrev.	Meaning
atto-	a-	10^{-18}	deka-	da-	10^{1}
femto-	f-	10^{-15}	hecto-	h-	10^{2}
pico-	p-	10^{-12}	kilo-	k-	10^{3}
nano-	n-	10^{-9}	mega-	M-	10^{6}
micro-	μ-	10^{-6}	giga-	G-	10^{9}
milli-	m-	10^{-3}	tera-	T-	10^{12}
centi-	c-	10^{-2}	peta-	P-	10^{15}
deci-	d-	10^{-1}	exa-	E-	10^{18}

The Uniqueness Theorem

Let us assume that we have two solutions of Laplace's equation, V_1 and V_2, both general functions of the coordinates used. Therefore

$$\nabla^2 V_1 = 0$$

and

$$\nabla^2 V_2 = 0$$

from which

$$\nabla^2 (V_1 - V_2) = 0$$

Each solution must also satisfy the boundary conditions, and if we represent the given potential values on the boundaries by V_b, then the value of V_1 on the boundary V_{1b} and the value of V_2 on the boundary V_{2b} must both be identical to V_b,

$$V_{1b} = V_{2b} = V_b$$

or

$$V_{1b} - V_{2b} = 0$$

In Section 4.8, Eq. (43), we made use of a vector identity,

$$\nabla \cdot (V\mathbf{D}) \equiv V(\nabla \cdot \mathbf{D}) + \mathbf{D} \cdot (\nabla V)$$

which holds for any scalar V and any vector $\mathbf{D}$. For the present application we shall select $V_1 - V_2$ as the scalar and $\nabla(V_1 - V_2)$ as the vector, giving

$$\nabla \cdot [(V_1 - V_2)\nabla(V_1 - V_2)] \equiv (V_1 - V_2)[\nabla \cdot \nabla(V_1 - V_2)] + \nabla(V_1 - V_2) \cdot \nabla(V_1 - V_2)$$

Coulomb's Law and Electric Field Intensity

Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

$$F = k\frac{Q_1 Q_2}{R^2}$$

where Q_1 and Q_2 are the positive or negative quantities of charge, R is the separation, and k is a proportionality constant. If the International System of Units[1] (SI) is used, Q is measured in coulombs (C), R is in meters (m), and the force should be newtons (N). This will be achieved if the constant of proportionality k is written as

$$k = \frac{1}{4\pi\epsilon_0}$$

The new constant ϵ_0 is called the *permittivity of free space* and has magnitude, measured in farads per meter (F/m),

$$\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi}10^{-9} \text{ F/m} \tag{1}$$

The quantity ϵ_0 is not dimensionless, for Coulomb's law shows that it has the label $C^2/N \cdot m^2$. We will later define the farad and show that it has the dimensions $C^2/N \cdot m$; we have anticipated this definition by using the unit F/m in equation (1).

Coulomb's law is now

$$F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2} \tag{2}$$

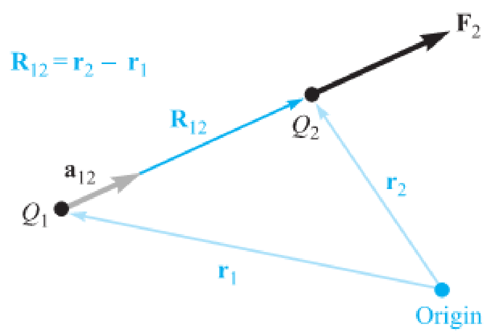


Figure 2.1 If Q_1 and Q_2 have like signs, the vector force F_2 on Q_2 is in the same direction as the vector R_{12}.

and is repulsive if the charges are alike in sign or attractive if they are of opposite sign. Let the vector $\mathbf{r}_1$ locate Q_1, whereas $\mathbf{r}_2$ locates Q_2. Then the vector $\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1$ represents the directed line segment from Q_1 to Q_2, as shown in Figure 2.1. The vector $\mathbf{F}_2$ is the force on Q_2 and is shown for the case where Q_1 and Q_2 have the same sign. The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} \tag{3}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of R_{12}, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|} \tag{4}$$

EXAMPLE 2.1

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

Solution. We use (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}$$

The magnitude of the force is 30 N, and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on Q_2 may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

D2.1. A charge $Q_A = -20\,\mu\text{C}$ is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\,\mu\text{C}$ is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$ (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) 8.854×10^{-12} F/m.

Ans. $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force *field* that is associated with charge, Q_1. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives the *electric field intensity*, $\mathbf{E}_1$ arising from Q_1:

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t} \qquad (6)$$

$\mathbf{E}_1$ is interpreted as the vector force, arising from charge Q_1, that acts on a unit positive test charge. More generally, we write the defining expression:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}} \qquad (7)$$

in which $\mathbf{E}$, a vector function, is the electric field intensity *evaluated at the test charge location* that arises from all *other* charges in the vicinity—meaning the electric field arising from the test charge itself is not included in $\mathbf{E}$.

The units of $\mathbf{E}$ would be in force per unit charge (newtons per coulomb). Again anticipating a new dimensional quantity, the *volt* (V), having the label of joules per

If we add more charges at other positions, the field due to n point charges is

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_m|^2}\mathbf{a}_m \qquad (11)$$

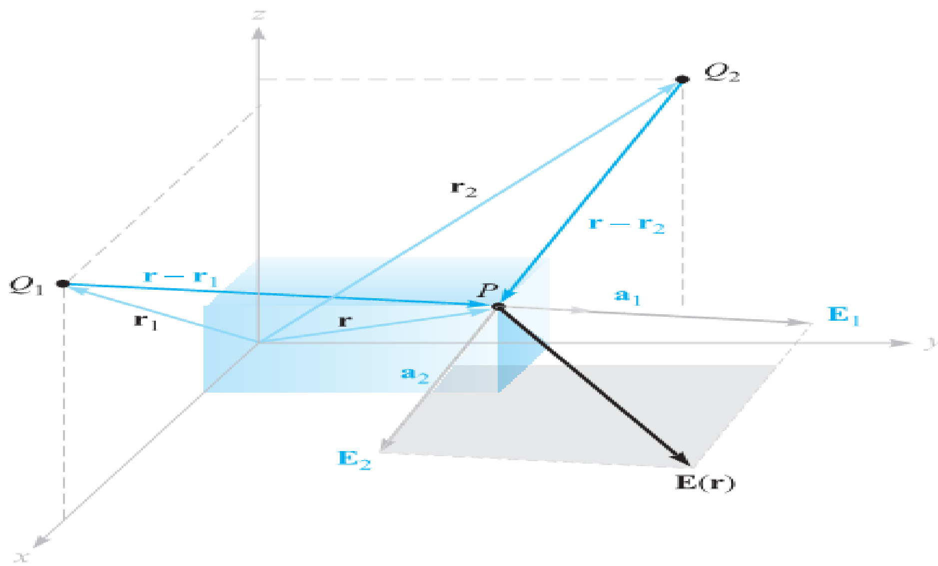


Figure 2.3 The vector addition of the total electric field intensity at P due to Q_1 and Q_2 is made possible by the linearity of Coulomb's law.

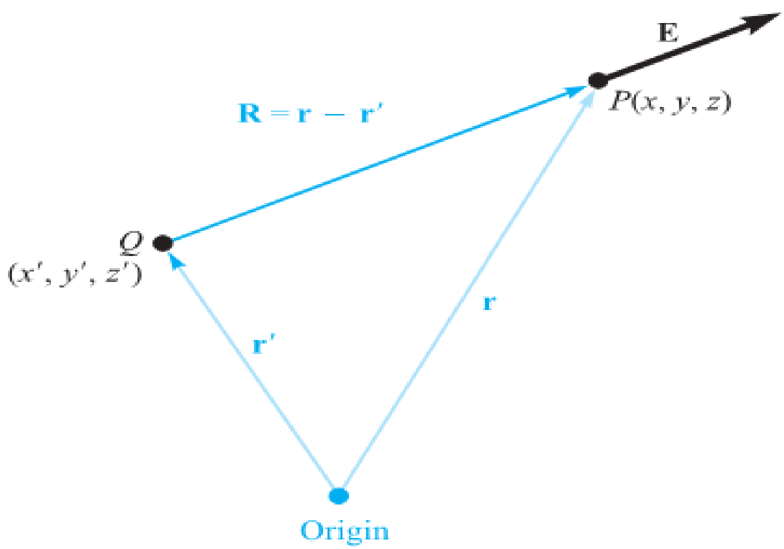


Figure 2.2 The vector $\mathbf{r}'$ locates the point charge Q, the vector $\mathbf{r}$ identifies the general point in space $P(x, y, z)$, and the vector $\mathbf{R}$ from Q to $P(x, y, z)$ is then $\mathbf{R} = \mathbf{r} - \mathbf{r}'$.

Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting **E** be symbolized in functional notation by **E**(**r**).

Because the coulomb forces are linear, the electric field intensity arising from two point charges, Q_1 at $\mathbf{r}_1$ and Q_2 at $\mathbf{r}_2$, is the sum of the forces on Q_t caused by Q_1 and Q_2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

EXAMPLE 2.2

In order to illustrate the application of (11), we find **E** at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r}-\mathbf{r}_1| = 1$, $|\mathbf{r}-\mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r}-\mathbf{r}_3| = 3$, and $|\mathbf{r}-\mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96\ \text{V} \cdot \text{m}$, we may now use (11) to obtain

$$\mathbf{E} = 26.96\left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2}\right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z\ \text{V/m}$$

D2.2. A charge of $-0.3\ \mu$C is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\ \mu$C is at $B(-10, 8, 12)$ cm. Find **E** at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

Ans. $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 94.2\mathbf{a}_z$ kV/m; $11.9\mathbf{a}_x - 0.519\mathbf{a}_y + 12.4\mathbf{a}_z$ kV/m

z

FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

We denote volume charge density by ρ_ν, having the units of coulombs per cubic meter (C/m^3).

The small amount of charge ΔQ in a small volume $\Delta \nu$ is

$$\Delta Q = \rho_\nu \Delta \nu \tag{12}$$

and we may define ρ_ν mathematically by using a limiting process on (12),

$$\rho_\nu = \lim_{\Delta \nu \to 0} \frac{\Delta Q}{\Delta \nu} \tag{13}$$

The total charge within some finite volume is obtained by integrating throughout that volume,

$$Q = \int_{\text{vol}} \rho_\nu d\nu \tag{14}$$

Only one integral sign is customarily indicated, but the differential $d\nu$ signifies integration throughout a volume, and hence a triple integration.

EXAMPLE 2.3

As an example of the evaluation of a volume integral, we find the total charge contained in a 2-cm length of the electron beam shown in Figure 2.5.

Solution. From the illustration, we see that the charge density is

$$\rho_\nu = -5 \times 10^{-6} e^{-10^5 \rho z} \text{ C/m}^2$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_0^{2\pi} \int_0^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

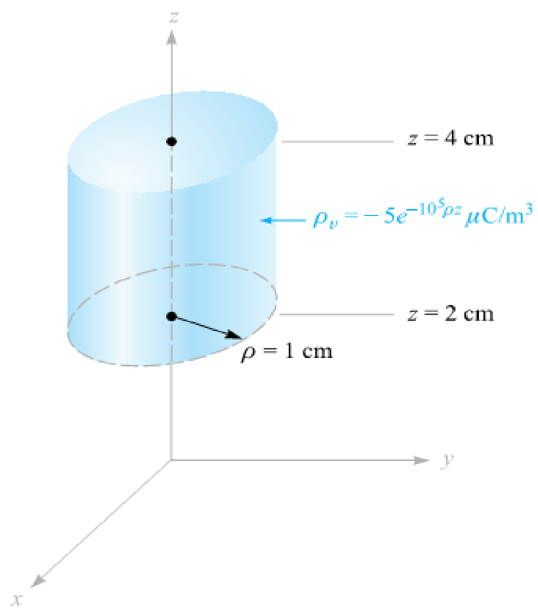
We integrate first with respect to ϕ because it is so easy,

$$Q = \int_{0.02}^{0.04} \int_0^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_0^{0.01} \left(\frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_0^{0.01} -10^{-5} \pi (e^{-2000\rho} - e^{-4000\rho}) d\rho$$



Finally,

$$Q = -10^{-10}\pi \left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000} \right)_0^{0.01}$$

$$Q = -10^{-10}\pi \left(\frac{1}{2000} - \frac{1}{4000} \right) = \frac{-\pi}{40} = 0.0785 \text{ pC}$$

D2.4. Calculate the total charge within each of the indicated volumes: $(a)\ 0.1 \leq |x|, |y|, |z| \leq 0.2$: $\rho_v = \dfrac{1}{x^3 y^3 z^3}$; $(b)\ 0 \leq \rho \leq 0.1, 0 \leq \phi \leq \pi, 2 \leq z \leq 4$; $\rho_v = \rho^2 z^2 \sin 0.6\phi$; (c) universe: $\rho_v = e^{-2r}/r^2$.

Ans. 0; 1.018 mC; 6.28 C

GAUSS'S LAW

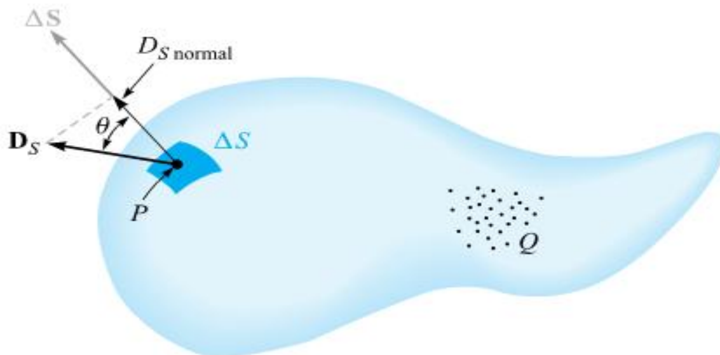


Figure 3.2 The electric flux density $\mathbf{D}_S$ at P arising from charge Q. The total flux passing through ΔS is $\mathbf{D}_S \cdot \Delta \mathbf{S}$.

where we are able to apply the definition

The *total* flux passing through the cl

ferential contributions crossing each surf

$$\Psi = \int d\Psi =$$

$$\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q \qquad (5)$$

or a volume charge distribution,

$$Q = \int_{\text{vol}} \rho_v \, dv$$

The last form is usually used, and we should agree now that it represents any or all of the other forms. With this understanding, Gauss's law may be written in terms of the charge distribution as

$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv \qquad (6)$$

EXAMPLE 3.1

To illustrate the application of Gauss's law, let us check the results of Faraday's experiment by placing a point charge Q at the origin of a spherical coordinate system (Figure 3.3) and by choosing our closed surface as a sphere of radius a.

Solution. We have, as before,

$$\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r$$

At the surface of the sphere,

$$\mathbf{D}_S = \frac{Q}{4\pi a^2}\mathbf{a}_r$$

The differential element of area on a spherical surface is, in spherical coordinates from Chapter 1,

$$dS = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi$$

or

$$d\mathbf{S} = a^2 \sin\theta \, d\theta \, d\phi \, \mathbf{a}_r$$

The integrand is

$$\mathbf{D}_S \cdot d\mathbf{S} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \mathbf{a}_r \cdot \mathbf{a}_r = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

leading to the closed surface integral

$$\int_{\phi=0}^{\phi=2\pi} \int_{\theta=\phi}^{\theta=\pi} \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\int_0^{2\pi} \frac{Q}{4\pi}(-\cos\theta)_0^{\pi} d\phi = \int_0^{2\pi} \frac{Q}{2\pi} d\phi = Q$$

D3.3. Given the electric flux density, $\mathbf{D} = 0.3r^2\mathbf{a}_r$ nC/m^2 in free space: (a) find $\mathbf{E}$ at point $P(r = 2, \theta = 25°, \phi = 90°)$; ($b$) find the total charge within the sphere $r = 3$; (c) find the total electric flux leaving the sphere $r = 4$.

Ans. 135.5$\mathbf{a}_r$ V/m; 305 nC; 965 nC

The Steady Magnetic Field

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We will find it necessary to accept several laws temporarily on faith alone. The proof of the laws does exist and is available on the Web site for the disbelievers or the more advanced student. ■

BIOT-SAVART LAW

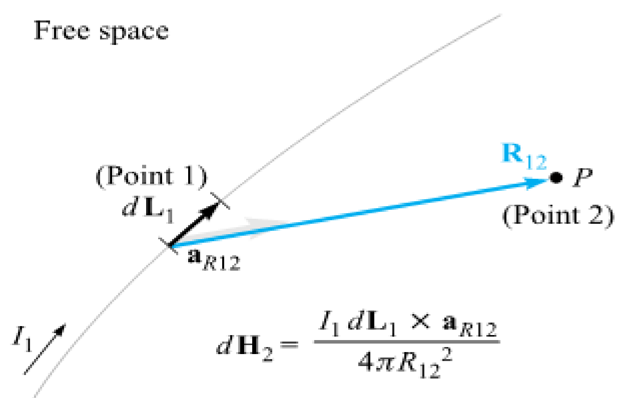


Figure 7.1 The law of Biot-Savart expresses the magnetic field intensity $d\mathbf{H}_2$ produced by a differential current element $I_1 d\mathbf{L}_1$. The direction of $d\mathbf{H}_2$ is into the page.

The units of the *magnetic field intensity* **H** are evidently amperes per meter (A/m). The geometry is illustrated in Figure 7.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} \tag{2}$$

The Biot-Savart law may also be expressed in terms of distributed sources, such as current density **J** and *surface current density* **K**. Surface current flows in a sheet of vanishingly small thickness, and the current density **J**, measured in amperes per square

meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by **K**. If the surface current density is uniform, the total current I in any width b is

$$I = Kb$$

where we assume that the width b is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Figure 7.2. For a nonuniform surface current density, integration is necessary:

$$I = \int K dN \tag{4}$$

where dN is a differential element of the path *across* which the current is flowing. Thus the differential current element $I\,d\mathbf{L}$, where $d\mathbf{L}$ is in the direction of the current, may be expressed in terms of surface current density **K** or current density **J**,

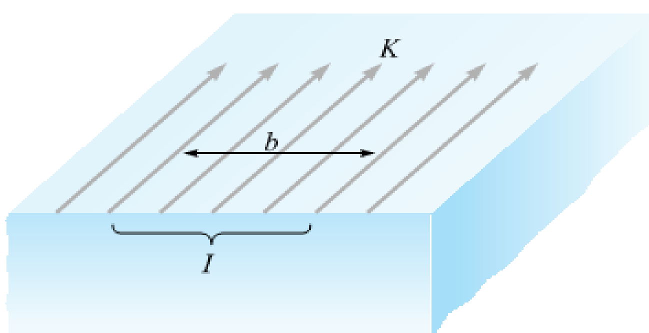


Figure 7.2 The total current I within a transverse width b, in which there is a *uniform* surface current density K, is Kb.

where dN is a differential element of the path *across* which the current is flowing. Thus the differential current element $I\,d\mathbf{L}$, where $d\mathbf{L}$ is in the direction of the current, may be expressed in terms of surface current density $\mathbf{K}$ or current density $\mathbf{J}$,

$$I\,d\mathbf{L} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{5}$$

and alternate forms of the Biot-Savart law obtained,

$$\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R\,dS}{4\pi R^2} \tag{6}$$

and

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R\,dv}{4\pi R^2} \tag{7}$$

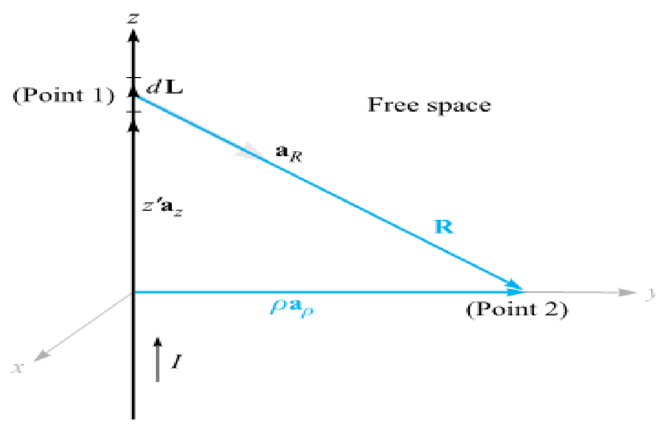


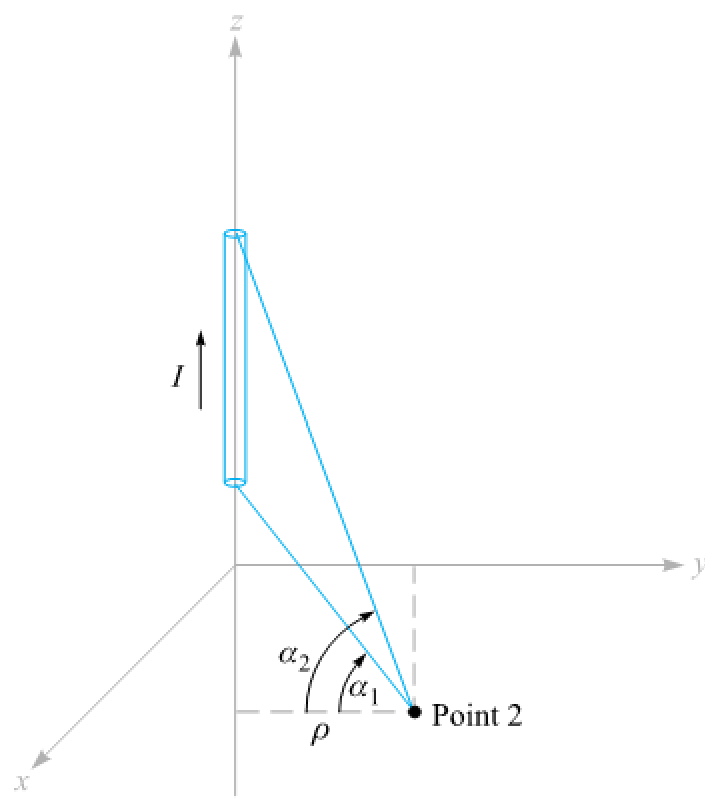
Figure 7.3 An infinitely long straight filament carrying a direct current I. The field at point 2 is $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$.

$$\mathbf{H}_2 = \frac{I}{2\pi\rho}\mathbf{a}_\phi \tag{8}$$

H is most easily expressed in terms of the angles α_1 and α_2, as identified in the figure. The result is

$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi \qquad (9)$$

If one or both ends are below point 2, then α_1 is or both α_1 and α_2 are negative.



EXAMPLE 7.1

As a numerical example illustrating the use of (9), we determine **H** at $P_2(0.4, 0.3, 0)$ in the field of an 8. A filamentary current is directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in Figure 7.6.

Solution. We first consider the semi-infinite current on the x axis, identifying the two angles, $\alpha_{1x} = -90°$ and $\alpha_{2x} = \tan^{-1}(0.4/0.3) = 53.1°$. The radial distance ρ is measured from the x axis, and we have $\rho_x = 0.3$. Thus, this contribution to $\mathbf{H}_2$ is

$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1° + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector $\mathbf{a}_\phi$ must also be referred to the x axis. We see that it becomes $-\mathbf{a}_z$. Therefore,

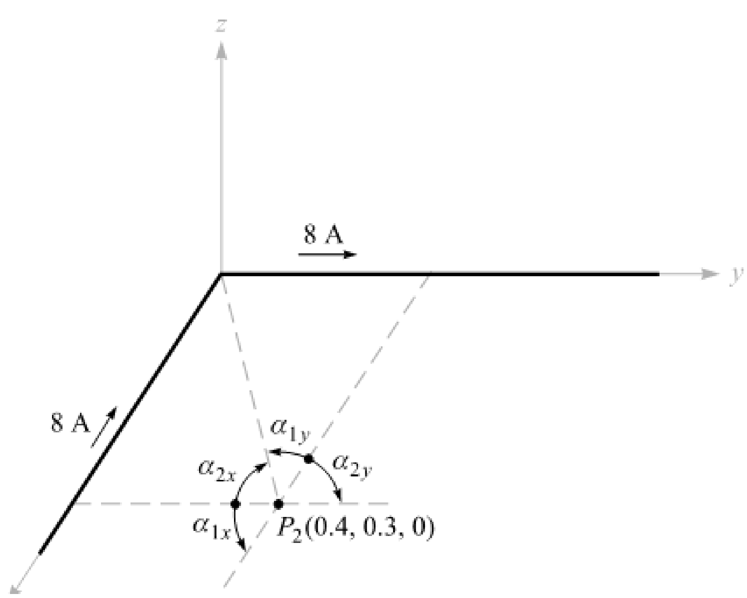
$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \text{ A/m}$$

For the current on the y axis, we have $\alpha_{1y} = -\tan^{-1}(0.3/0.4) = -36.9°$, $\alpha_{2y} = 90°$, and $\rho_y = 0.4$. It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9°)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \text{ A/m}$$

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi}\mathbf{a}_z = -6.37\mathbf{a}_z \text{ A/m}$$



AMPÈRE'S CIRCUITAL LAW

Ampère's circuital law states that the line integral of **H** about any *closed* path is exactly equal to the direct current enclosed by that path,

$$\oint \mathbf{H} \cdot d\mathbf{L} = I \qquad (10)$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

In our example, the path must be a circle of radius ρ, and Ampère's circuital law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

CURL

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampère's circuital law to the perimeter of a differential surface element and discuss the third and last of the special derivatives of vector analysis, the curl. Our objective is to obtain the point form of Ampère's circuital law.

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} \qquad (21)$$

In rectangular coordinates, the definition

This result may be written in the form of a determinant,

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

and may also be written in terms of the vector operator,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H}$$

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)} \tag{25}$$

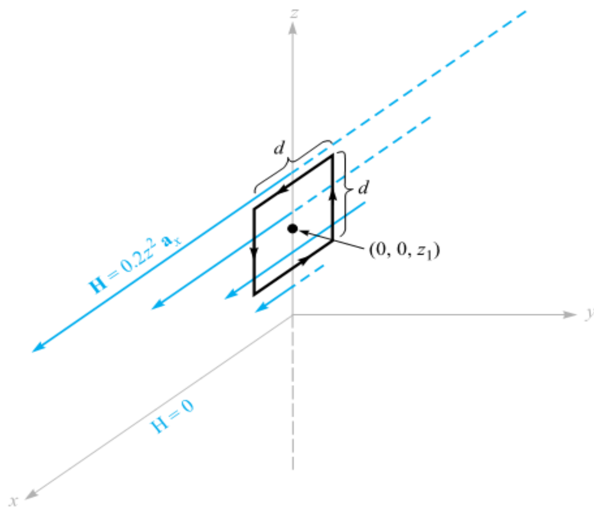
$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)} \tag{26}$$

Examples

1- Cart.
2- Cylind.
3- Sphere.

EXAMPLE 7.2

As an example of the evaluation of curl **H** from the definition and of the evaluation of another line integral, suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Figure 7.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > d/2$.



Solution. We evaluate the line integral of **H** along the four segments, beginning at the top:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0.2\left(z_1 + \tfrac{1}{2}d\right)^2 d + 0 - 0.2\left(z_1 - \tfrac{1}{2}d\right)^2 d + 0$$
$$= 0.4z_1 d^2$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d \to 0} \frac{0.4z_1 d^2}{d^2} = 0.4z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4z_1\mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the preceding result when $z = z_1$.

STOKES' THEOREM

because every *interior* wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore we have

$$\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} \tag{30}$$

where $d\mathbf{L}$ is taken only on the perimeter of S.

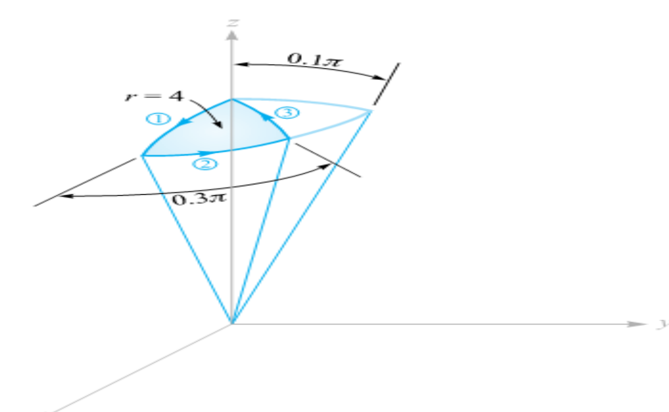
Equation (30) is an identity, holding for any vector field, and is known as *Stokes' theorem*.

EXAMPLE 7.3

A numerical example may help to illustrate the geometry involved in Stokes' theorem. Consider the portion of a sphere shown in Figure 7.17. The surface is specified by $r = 4, 0 \le \theta \le 0.1\pi, 0 \le \phi \le 0.3\pi$, and the closed path forming its perimeter is composed of three circular arcs. We are given the field $\mathbf{H} = 6r \sin\phi \mathbf{a}_r + 18r \sin\theta \cos\phi \mathbf{a}_\phi$ and are asked to evaluate each side of Stokes' theorem.

Solution. The first path segment is described in spherical coordinates by $r = 4, 0 \le \theta \le 0.1\pi, \phi = 0$; the second one by $r = 4, \theta = 0.1\pi, 0 \le \phi \le 0.3\pi$; and the third by $r = 4, 0 \le \theta \le 0.1\pi, \phi = 0.3\pi$. The differential path element $d\mathbf{L}$ is the vector sum of the three differential lengths of the spherical coordinate system first discussed in Section 1.9,

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi$$



The first term is zero on all three segments of the path since $r = 4$ and $dr = 0$, the second is zero on segment 2 as θ is constant, and the third term is zero on both segments 1 and 3. Thus,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_1 H_\theta r\, d\theta + \int_2 H_\phi r \sin\theta\, d\phi + \int_3 H_\theta r\, d\theta$$

Because $H_\theta = 0$, we have only the second integral to evaluate,

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{0.3\pi} [18(4) \sin 0.1\pi \cos\phi] 4 \sin 0.1\pi\, d\phi$$
$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

We next attack the surface integral. First, we use (26) to find

$$\nabla \times \mathbf{H} = \frac{1}{r \sin\theta}(36r \sin\theta \cos\theta \cos\phi)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta} 6r \cos\phi - 36r \sin\theta \cos\phi\right)\mathbf{a}_\theta$$

Because $d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{a}_r$, the integral is

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_0^{0.3\pi} \int_0^{0.1\pi} (36 \cos\theta \cos\phi) 16 \sin\theta\, d\theta\, d\phi$$
$$= \int_0^{0.3\pi} 576 \left(\tfrac{1}{2} \sin^2\theta\right)\Big|_0^{0.1\pi} \cos\phi\, d\phi$$
$$= 288 \sin^2 0.1\pi \sin 0.3\pi = 22.2 \text{ A}$$

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A

MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the *magnetic flux density* **B** as

$$\mathbf{B} = \mu_0 \mathbf{H} \quad \text{(free space only)} \tag{32}$$

where **B** is measured in webers per square meter (Wb/m^2) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1Wb/m^2 is the same as 10, 000 G. The constant μ_0 is not dimensionless and has the *defined value* for free space, in henrys per meter (H/m), of

$$\mu_0 = 4\pi \times 10^{-7} \text{ H/m} \tag{33}$$

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current I, the **H** field formed concentric circles about the filament. Because $\mathbf{B} = \mu_0\mathbf{H}$, the **B** field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge." For this reason Gauss's law for the magnetic field is

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \tag{35}$$

and application of the divergence theorem shows us that

$$\nabla \cdot \mathbf{B} = 0 \tag{36}$$

Equation (36) is the last of Maxwell's four equations as they apply to static electric fields and steady magnetic fields. Collecting these equations, we then have for static electric fields and steady magnetic fields

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned} \tag{37}$$

To these equations we may add the two expressions relating **D** to **E** and **B** to **H** in free space,

$$\mathbf{D} = \epsilon_0 \mathbf{E} \tag{38}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \tag{39}$$

We have also found it helpful to define an electrostatic potential,

$$\mathbf{E} = -\nabla V \tag{40}$$

The corresponding set of four integral

equations that apply to static electric fields and steady magnetic fields is

$$\begin{aligned}
&\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v \, dv \\
&\oint \mathbf{E} \cdot d\mathbf{L} = 0 \\
&\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S} \\
&\oint_S \mathbf{B} \cdot d\mathbf{S} = 0
\end{aligned} \tag{41}$$

As an example of the use of flux and flux density in magnetic fields, let us find the flux between the conductors of the coaxial line of Figure 7.8a. The magnetic field intensity was found to be

$$H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$$

and therefore

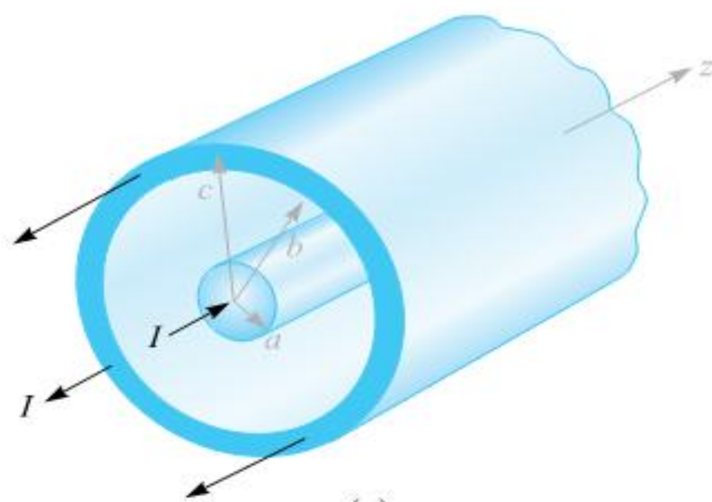
$$\mathbf{B} = \mu_0 \mathbf{H} = \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi$$

The magnetic flux contained between the conductors in a length d is the flux crossing any radial plane extending from $\rho = a$ to $\rho = b$ and from, say, $z = 0$ to $z = d$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\mathbf{a}_\phi \cdot d\rho\, dz\, \mathbf{a}_\phi$$

or

$$\Phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \tag{42}$$



(a)

D7.7. A solid conductor of circular cross section is made of a homogeneous nonmagnetic material. If the radius $a = 1$ mm, the conductor axis lies on the z axis, and the total current in the $\mathbf{a}_z$ direction is 20 A, find: (*a*) H_ϕ at $\rho = 0.5$ mm; (*b*) B_ϕ at $\rho = 0.8$ mm; (*c*) the total magnetic flux per unit length inside the conductor; (*d*) the total flux for $\rho < 0.5$ mm; (*e*) the total magnetic flux outside the conductor.

Ans. 1592 A/m; 3.2 mT; 2 μWb/m; 0.5 μWb; ∞

Examples (1- Cart. 2- Cylind. 3- Spher.)

Time-Varying Fields and Maxwell's Equations

In the previous chapters we have studied the basic concepts in an electrostatic and magnetostatic fields. These fields can be considered as time invariant or static fields. In static electromagnetic fields, electric and magnetic fields are independent of each other. In this chapter, we shall concentrate on the time varying or dynamic fields. In dynamic electromagnetic fields, the electric and magnetic fields are interdependent. In general, static electric fields are produced by stationary electric charges. The static magnetic fields are produced due to the motion of the electric charges with uniform velocity or the magnetic charges. The time varying fields are produced due to the time varying currents.

Faraday's Law

According to Faraday's experiment, a static magnetic field cannot produce any current flow. But with a time varying field, an electromotive force (e.m.f.) induces which may drive a current in a closed path or circuit. This e.m.f. is nothing but a voltage that induces from changing magnetic fields or motion of the conductors in a magnetic field. Faraday discovered that the induced e.m.f. is equal to the time rate of change of magnetic flux linking with the closed circuit.

Faraday's law can be stated as,

$$\boxed{e = -N\frac{d\phi}{dt} \text{ volts.}} \quad \text{... (1)}$$

where N = Number of turns in the circuit

e = Induced e.m.f.

Let us assume single turn circuit i.e. $N = 1$, then Faraday's law can be stated as,

$$e = -\frac{d\phi}{dt} \text{ volts} \quad \text{... (2)}$$

Let us consider Faraday's law. The induced e.m.f. is a scalar quantity measured in volts. Thus the induced e.m.f. is given by,

$$\boxed{e = \oint \bar{E} \cdot d\bar{L}} \quad \text{... (3)}$$

The induced e.m.f. in equation (3) indicates a voltage about a closed path such that if any part of the path is changed, the e.m.f. will also change.

The magnetic flux ϕ passing through a specified area is given by,

$$\phi = \int_S \bar{B} \cdot d\bar{S}$$

where B = Magnetic flux density

Using above result, equation (2) can be rewritten as,

$$e = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{S} \quad \text{... (4)}$$

From equations (3) and (4), we get,

$$\boxed{e = \oint \bar{E} \cdot d\bar{L} = -\frac{d}{dt}\int_S \bar{B} \cdot d\bar{S}} \quad \text{... (5)}$$

There are two conditions for the induced e.m.f. as explained below.

i) The closed circuit in which e.m.f. is induced is stationary and the magnetic flux is sinusoidally varying with time. From equation (5) it is clear that the magnetic flux density is the only quantity varying with time. We can use partial derivative to define relationship as $\overline{B}$ may be changing with the co-ordinates as well as time. Hence we can write,

$$\oint \overline{E} \cdot d\overline{L} = -\int_S \frac{\partial \overline{B}}{\partial t} \cdot d\overline{S} \qquad \text{... (6)}$$

This is similar to **transformer action** and e.m.f. is called **transformer e.m.f.** Using Stoke's theorem, a line integral can be converted to the surface integral as

$$\oint_S (\nabla \times \overline{E}) \cdot d\overline{S} = -\int_S \frac{\partial \overline{B}}{\partial t} \cdot d\overline{S} \qquad \text{... (7)}$$

Assuming that both the surface integrals taken over identical surfaces.

$$\therefore \quad (\nabla \times \overline{E}) \cdot d\overline{S} = -\frac{\partial \overline{B}}{\partial t} \cdot d\overline{S}$$

Hence finally,

$$\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \qquad \text{... (8)}$$

Equation (8) represents one of the Maxwell's equations. If $\overline{B}$ is not varying with time, then equations (6) and (8) give the results obtained previously in the electrostatics.

$$\oint \overline{E} \cdot d\overline{L} = 0, \text{ and}$$

$$\nabla \times \overline{E} = 0$$

ii) Secondly magnetic field is stationary, constant not varying with time while the closed circuit is revolved to get the relative motion between them. This action is similar to **generator action**, hence the induced e.m.f. is called **motional or generator e.m.f.**

Consider that a charge Q is moved in a magnetic field $\overline{B}$ at a velocity $\overline{v}$. Then the force on a charge is given by,

$$\overline{F} = Q\,\overline{v} \times \overline{B} \qquad \text{... (9)}$$

But the motional electric field intensity is defined as the force per unit charge. It is given by,

$$\therefore \quad \overline{E}_m = \frac{\overline{F}}{Q} = \overline{v} \times \overline{B} \qquad \text{... (10)}$$

Thus the induced e.m.f. is given by,

$$\oint \overline{E}_m \cdot d\overline{L} = \oint (\overline{v} \times \overline{B}) \cdot d\overline{L} \qquad \text{... (11)}$$

Equation (11) represents total e.m.f. induced when a conductor is moved in a uniform constant magnetic field.

If the directions of velocity $\overline{v}$ with which conductor is moving and the magnetic field $\overline{B}$ are mutually perpendicular to each other, then the induced e.m.f. is given by,

$$e = Blv \sin 90° = Blv \qquad \text{..(12)}$$

where l = Length of straight conductor

iii) If in case, the magnetic flux density is also varying with time, then the induced e.m.f. is the combination of transformer e.m.f. and generator e.m.f. given by,

$$\oint \bar{E}\cdot d\bar{L} = -\int \frac{\partial \bar{B}}{\partial t}\cdot d\bar{S}+\oint (\bar{v}\times\bar{B})\cdot d\bar{L} \quad \text{... (13)}$$

Example 9.1 : *A conductor 1 cm in length is parallel to z-axis and rotates at radius of 25 cm at 1200 r.p.m. Find induced voltage, if the radial field is given by,*

$\bar{B} = 0.5\bar{a}_r$ T

Solution : In above case, the magnetic flux is constant while the path is rotating at 1200 r.p.m. Under such condition, the field intensity is given by,

$$\bar{E} = \bar{v}\times\bar{B}$$

where $\bar{v}$ = Linear velocity

In 1 minute there are 1200 revolutions which corresponds to 20 revolutions in one second. In one revolution distance travelled is $(2\pi r)$ meter. Hence in 20 revolutions the distance travelled in one second is $(40\pi r)$ meter. The conductor rotates in ϕ-direction. Hence linear velocity is given by,

$$\bar{v} = (40\pi r)\bar{a}_\phi = 40\pi(25\times10^{-2})\bar{a}_\phi = 31.416\,\bar{a}_\phi \text{ m/s}$$

Hence an electric field intensity is calculated as,

$$\bar{E} = [31.416\bar{a}_\phi]\times[0.5\bar{a}_r] = 15.708\,(-\bar{a}_z) \quad \text{... } \bar{a}_\phi\times\bar{a}_r = -\bar{a}_z$$

Induced voltage is given by,

$$e = \oint \bar{E}\cdot d\bar{L}$$

Now $d\bar{L} = (dz)\bar{a}_z$ as conductor is parallel to z-axis.

$$e = \int_{z=0}^{0.01} 15.708(-\bar{a}_z)\cdot(dz)\bar{a}_z$$

$$= -15.708[z]_0^{0.01} = -157.08 \text{ mV}$$

Examples (1- Cart. 2- Spher. 3- Cylind.)

General Field Relations for Time Varying Electric and Magnetic Fields

The basic relation between an electric and magnetic field, starting from Faraday's law is given by,

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} \quad \text{... (1)}$$

But we have already studied that,

$$\bar{B} = \nabla \times \bar{A}$$ where $\bar{A}$ is vector magnetic potential.

$$\therefore \quad \nabla \times \bar{E} = -\frac{\partial}{\partial t}\left(\nabla \times \bar{A}\right) \quad \text{... (2)}$$

Intercharging operators at R.H.S. of above equation, we get,

$$\nabla \times \bar{E} = -\nabla \times \frac{\partial \bar{A}}{\partial t}$$

$$\therefore \quad \nabla \times \bar{E} + \nabla \times \frac{\partial \bar{A}}{\partial t} = 0$$

$$\therefore \quad \nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = 0 \quad \text{... (3)}$$

But according to vector identity **'curl' of a gradient of a scalar is always zero'**. Hence we can write,

$$\bar{E} + \frac{\partial \bar{A}}{\partial t} = \nabla V \quad \text{... (4)}$$

As R.H.S. of the equation (3) including curl is zero, we can introduce negative sign at R.H.S. of the equation (4).

$$\therefore \quad \bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t} \quad \text{... (5)}$$

Now when the field is static, $\frac{\partial \bar{A}}{\partial t} = 0$, hence we get basic gradient relationship as,

$$\bar{E} = -\nabla V \quad \text{... (6)}$$

Consider any closed surface. If the current is flowing out of the surface, we can write

$$I = \frac{dQ}{dt} \text{ A i.e. C/sec}$$

As current is flowing out of the surface, it indicates that positive charge is going out. So the positive charge is decreasing internally. Let Q_I be the internal charge,

$$\therefore \quad I = -\frac{dQ_I}{dt} \quad \text{... (7)}$$

If there is a volume charge ρ_v, then we can write,

$$Q_I = \int_v \rho_v \, dv \quad \text{... (8)}$$

$$\therefore \quad I = -\frac{d}{dt}\left[\int_v \rho_v \, dv\right]$$

Changing operations, we can write,

$$I = -\int_v \frac{d\rho_v}{dt}\, dv \qquad \text{... (9)}$$

But current can be expressed as

$$I = \int_v \bar{J} \cdot d\bar{S} \qquad \text{... (10)}$$

Equating equations (9) and (10),

$$\int_v \bar{J} \cdot d\bar{S} = -\int_v \frac{d\rho_v}{dt}\, dv$$

Using divergence theorem, converting surface integral to volume integral, assuming that the volume v is enclosed by the same surface S.

$$\therefore \quad \int_v \nabla \cdot J\, dv = -\int_V \frac{d\rho_v}{dt}\, dv$$

$$\therefore \quad \boxed{\nabla \cdot \bar{J} = -\frac{d\rho_v}{dt}} \qquad \text{... (11)}$$

Equation (11) is called **equation of continuity of current in point or differential form**

Consider Ampere's circuit law in point or differential form as,

$$\nabla \times \bar{H} = \bar{J}$$

Taking divergence on both sides of above equation, we get,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} \qquad \text{... (12)}$$

According to vector identity, **'divergence of curl of vector is zero'.**

But $\nabla \cdot \bar{J} = 0$ is valid only for static fields. For time varying field, we must modify above relation to have above property valid as,

$$\nabla \cdot \bar{J} = -\frac{\partial \rho_v}{\partial t}$$

$$\therefore \quad \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t} = 0 \qquad \text{... (13)}$$

Now for time varying fields we can write,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \frac{\partial \rho_v}{\partial t} \qquad \text{...(14)}$$

But we know that,

$$\nabla \cdot \bar{D} = \rho_v \qquad \text{... Gauss's law in point form,}$$

Putting in equation (14),

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \frac{\partial}{\partial t}[\nabla \cdot \bar{D}]$$

Interchanging operations of R.H.S. of above equation,

$$\nabla \cdot (\nabla \times \bar{H}) = \nabla \cdot \bar{J} + \nabla \cdot \frac{\partial \bar{D}}{\partial t}$$

or

$$\boxed{\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}} \quad \ldots (15)$$

Above equation is Ampere's circuit law for time varying fields. In this equation, $\bar{J}$ represents conduction current density while $\frac{\partial \bar{D}}{\partial t}$ represents displacement current density. So we can rewrite equation (15) as follows.

$$\boxed{\nabla \times \bar{H} = \bar{J}_C + \bar{J}_D} \quad \ldots (16)$$

Maxwell's Equations

Differential form	Integral form	Significance
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \cdot d\bar{L} = -\int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{S}$	Faraday's law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \cdot d\bar{L} = I + \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{S}$	Ampere's circuital law
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{S} = \int_S \rho_v \, dv$	Gauss's law
$\nabla \cdot \bar{B} = 0$	$\oint_S \bar{B} \cdot d\bar{S} = 0$	No isolated magnetic charges.

Maxwell's Equations for Free Space

In the previous section, we have obtained Maxwell's equations in integral and point form. Let us consider now free space as a medium in which fields are present. Free space is a non-conducting medium in which volume charge density ρ_v is zero and conductivity σ is also zero.

The Maxwell's equation, in the free space are as mentioned below.

A) Point Form :

i) $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

Example 9.5 : *If the magnetic field $\bar{H} = [3x\cos\beta + 6y\sin\alpha]\bar{a}_z$, find current density $\bar{J}$ if fields are invariant with time.*

Solution : The point form of Maxwell's second equation is,

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$$

But as fields are time invariant, we can write,

$$\frac{\partial \bar{D}}{\partial t} = 0$$

$$\therefore \quad \nabla \times \bar{H} = \bar{J}$$

$$\therefore \quad \bar{J} = \begin{vmatrix} \bar{a}_x & \bar{a}_y & \bar{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & (3x\cos\beta + 6y\sin\alpha) \end{vmatrix}$$

$$\therefore \quad \bar{J} = \frac{\partial}{\partial y}[3x\cos\beta + 6y\sin\alpha]\bar{a}_x - \frac{\partial}{\partial x}[3x\cos\beta + 6y\sin\alpha]\bar{a}_y$$

$$\therefore \quad \bar{J} = 6\sin\alpha\,\bar{a}_x - 3\cos\beta\,\bar{a}_y \quad A/m^2$$

In other words, if the ratio of the magnitudes of the current densities is greater than 1, the medium is conductor and if the ratio of the magnitudes is less than 1 then the medium is dielectric

If	$\frac{\sigma}{\omega\varepsilon} >> 1,$	Medium is conductor
If	$\frac{\sigma}{\omega\varepsilon} << 1,$	Medium is dielectric

Also the ratio represented above depends on frequency, a medium which is conductor at low frequency may become insulator at very high frequency.

For lossy dielectric medium,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}$$

Example 9.3 : *In a given lossy dielectric medium, conduction current density $J_C = 0.02 \sin 10^9 t \, (A/m^2)$. Find the displacement current density if $\sigma = 10^3$ S/m and $\varepsilon_r = 6.5$.*

Solution : For lossy dielectric medium,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon}$$

$$\therefore \quad J_D = \frac{\omega\varepsilon J_C}{\sigma} = \frac{10^9 \times (\varepsilon_r \varepsilon_0) \times 0.02}{10^3}$$

$$\therefore \quad J_D = \frac{10^9 \times 6.5 \times 8.854 \times 10^{-12} \times 0.02}{10^3}$$

$$\therefore \quad J_D = 1.151 \times 10^{-6} \text{ A/m}^2 = \mathbf{1.151 \ \mu A/m^2}$$

As $\bar{J}_D$ and $\bar{J}_C$ are always at right angles to each other, we can write,

$$\bar{J}_D = \mathbf{1.151 \cos 10^9 t \ \mu A/m^2}$$

Example 9.11 : *Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2 \times 10^{-4}$ ℧/m and $\varepsilon_r = 81$.*

Solution : The ratio of amplitudes of the two current densities is given as 1, so we can write,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = 1$$

i.e. $$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

$$\therefore \quad \omega = \frac{2 \times 10^{-4}}{(8.854 \times 10^{-12})(81)} = 0.2788 \times 10^6 \text{ rad/sec}$$

But $$\omega = 2\pi f$$

$$\therefore \quad f = \frac{\omega}{2\pi} = \frac{0.2788 \times 10^6}{2\pi} = \mathbf{44.372 \ kHz}$$

Example 9.12 : *In a material for which $\sigma = 5.0$ S/m and $\varepsilon_r = 1$, the electric field intensity is $E = 250 \sin 10^{10} t$ V/m. Find the conduction and displacement current densities, and the frequency at which both have equal magnitudes.*

Solution : The conduction current density is given by,

$$J_C = \sigma E$$
$$= 5\left(250 \sin 10^{10} t\right)$$
$$= 1250 \sin 10^{10} t \quad A/m^2$$

The displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\varepsilon E)$$
$$= \frac{\partial}{\partial t}[\varepsilon_0 \varepsilon_r E]$$
$$= \frac{\partial}{\partial t}\left[8.854 \times 10^{-12} \times 1 \times 250 \sin 10^{10} t\right]$$
$$= \left(8.854 \times 10^{-12} \times 250\right)\left(10^{10}\right)\left(\cos 10^{10} t\right)$$
$$= 22.135 \cos 10^{10} t \quad A/m^2$$

For the two densities, the condition for magnitudes to be equal is,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\varepsilon \omega} = 1$$

Example 9.15 : *Find the amplitude of the displacement current density,*
a) In the air near car antenna where the field strength of FM signal is,
$\bar{E} = 80\cos\left(6.277\times10^{8}t - 2.092y\right)\bar{a}_z$ *V/m ;*
b) Inside a capacitor where $\varepsilon_r = 600$ *and*
$\bar{D} = 3\times10^{-6}\,\sin\left(6\times10^{6}t - 0.3464x\right)\bar{a}_z$ *C/m²*.

Solution : $\bar{E} = 80\cos\left(6.277\times10^{8}t - 2.092y\right)\bar{a}_z$

The displacement current density is given by,

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon_0\varepsilon_r\bar{E}\right)$$

For air, $\varepsilon_r = 1$

$$\therefore \quad \bar{J}_D = \varepsilon_0\frac{\partial \bar{E}}{\partial t}$$

$$= \varepsilon_0\frac{\partial}{\partial t}\left[80\cos\left(6.277\times10^{8}t - 2.092y\right)\bar{a}_z\right]$$

$$= \left(8.854\times10^{-12}\right)(80)\left(-6.277\times10^{8}\right)\sin\left(6.277\times10^{8}t - 2.092y\right)\bar{a}_z$$

$$\therefore \quad \bar{J}_D = -0.4446\sin\left(6.277\times10^{8}t - 2.092y\right)\bar{a}_z \quad \text{A/m}^2$$

Thus the amplitude of the displacement current density is,

$$J_D = 0.4446 \ \text{A/m}^2$$

b) Inside capacitor $\varepsilon_r = 600$, the displacement current density is given by,

$$\bar{J}_D = \frac{\partial \bar{D}}{\partial t}$$

$$= \frac{\partial}{\partial t}\left[3\times10^{-6}\sin\left(6\times10^{6}t - 0.3464x\right)\bar{a}_z\right]$$

$$= \left(3\times10^{-6}\right)\left(6\times10^{6}\right)\cos\left(6\times10^{6}t - 0.3464x\right)\bar{a}_z$$

$$= 18\cos\left(6\times10^{6}t - 0.3464x\right)\bar{a}_z \quad \text{A/m}^2$$

Hence the amplitude of displacement current density is,

$$J_D = 18 \ \text{A/m}^2$$

Example 9.27 : *A parallel plate capacitor with plate area of 5 cm^2 and plate separation of 3 mm has a voltage of 50 sin 10^3t Volts applied to its plates. Calculate the displacement current assuming $\varepsilon = 2\varepsilon_0$.*

Solution : $D = \varepsilon E = \varepsilon \frac{V}{d}$

Hence the displacement current density is given by,

$$J_D = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}\left(\varepsilon \frac{V}{d}\right)$$

$$= \frac{\varepsilon}{d}\frac{dV}{dt}$$

Hence the displacement current is given by

$$i_D = J_D \cdot \text{Area} = \left(\frac{\varepsilon}{d}\frac{dV}{dt}\right)(A) \qquad \text{... Plate area} = A$$

$$\therefore \quad i_D = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

This current is same as conduction current.

$$\therefore \quad i_C = \frac{dQ}{dt} = A\frac{dD}{dt} = \varepsilon A\frac{dE}{dt} = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

Hence the conduction current and displacement current is same. The displacement current is given by

$$i_D = \frac{\varepsilon A}{d}\frac{dV}{dt}$$

$$= \frac{(2\varepsilon_0)(A)}{d}\frac{dV}{dt}$$

$$= \frac{2\times 8.854\times 10^{-12}\times 5\times 10^{-4}}{3\times 10^{-3}}\frac{d}{dt}(50\sin 10^3 t)$$

$$= \frac{2\times 8.854\times 10^{-12}\times 5\times 10^{-4}\times 50\times 10^3}{3\times 10^{-3}}\cos 10^3 t$$

$$= \mathbf{0.1475 \cos 10^3 t \ \mu A}$$

Example 9.31 : *A No 10 copper wire carries a conduction current of 1 amp at 60 Hz. Calculate the displacement current in the wire. For copper assume,*

$$\varepsilon = \varepsilon_0 = \frac{1}{36\times\pi\times10^9}\ F/m = 8.854\times10^{-12}\ F/m$$

$$\mu = \mu_0 = 4\pi\times10^{-7}\ H/m$$

$$\sigma = 5.8\times10^7\ \mho/m$$

(UPTU : 2005-06)

Solution : By definition,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = \frac{5.8\times10^7}{2\times\pi\times60\times8.854\times10^{-12}} = 1.7376\times10^{16}$$

But $|\bar{J}_C| = \frac{i_C}{A}$ and $|\bar{J}_D| = \frac{i_D}{A}$

$$\therefore \quad \frac{i_C/A}{i_D/A} = 1.7376\times10^{16}$$

$$\therefore \quad i_D = \frac{i_C}{1.7376\times10^{16}} = \frac{1}{1.7376\times10^{16}} = 0.05755\times10^{-15}\ A$$

ELECTROMAGNETIC WAVE PROPAGATION

In general, **waves** are means of transporting energy or information.

Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity; in traveling, they assume the properties of waves; and they radiate outward from a source, without benefit of any discernible physical vehicles. The problem of radiation will be addressed in Chapter 13.

In this chapter, our major goal is to solve Maxwell's equations and derive EM wave motion in the following media:

1. Free space ($\sigma = 0, \varepsilon = \varepsilon_o, \mu = \mu_o$)
2. Lossless dielectrics ($\sigma = 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \ll \omega\varepsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \varepsilon = \varepsilon_r\varepsilon_o, \mu = \mu_r\mu_o$)
4. Good conductors ($\sigma \simeq \infty, \varepsilon = \varepsilon_o, \mu = \mu_r\mu_o$, or $\sigma \gg \omega\varepsilon$)

WAVES IN GENERAL

A clear understanding of EM wave propagation depends on a grasp of what waves are in general.

A **wave** is a function of both space and time.

Wave motion occurs when a disturbance at point A, at time t_o, is related to what happens at point B, at time $t > t_o$. A wave equation, as exemplified by eqs. (9.51) and (9.52), is a partial differential equation of the second order. In one dimension, a scalar wave equation takes the form of

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0 \tag{10.1}$$

If we particularly assume harmonic (or sinusoidal) time dependence $e^{j\omega t}$, eq. (10.1) becomes

$$\frac{d^2E_s}{dz^2} + \beta^2 E_s = 0 \tag{10.3}$$

where $\beta = \omega/u$ and E_s is the phasor form of E. The solution to eq. (10.3) is similar to Case 3 of Example 6.5 [see eq. (6.5.12)]. With the time factor inserted, the possible solutions to eq. (10.3) are

$$E^+ = Ae^{j(\omega t - \beta z)} \tag{10.4a}$$

$$E^- = Be^{j(\omega t + \beta z)} \tag{10.4b}$$

and

$$E = Ae^{j(\omega t - \beta z)} + Be^{j(\omega t + \beta z)} \tag{10.4c}$$

where A and B are real constants.

For the moment, let us consider the solution in eq. (10.4a). Taking the imaginary part of this equation, we have

$$E = A \sin(\omega t - \beta z) \tag{10.5}$$

Due to the variation of E with both time t and space variable z, we may plot E as a function of t by keeping z constant and vice versa. The plots of $E(z, t = \text{constant})$ and $E(t, z = \text{constant})$ are shown in Figure 10.1(a) and (b), respectively. From Figure 10.1(a), we observe that the wave takes distance λ to repeat itself and hence λ is called the *wavelength* (in meters). From Figure 10.1(b), the wave takes time T to repeat itself; consequently T is known as the *period* (in seconds). Since it takes time T for the wave to travel distance λ at the speed u, we expect

$$\lambda = uT \tag{10.6a}$$

But $T = 1/f$, where f is the *frequency* (the number of cycles per second) of the wave in Hertz (Hz). Hence,

$$\boxed{u = f\lambda} \tag{10.6b}$$

Because of this fixed relationship between wavelength and frequency, one can identify the position of a radio station within its band by either the frequency or the wavelength. Usually the frequency is preferred. Also, because

$$\omega = 2\pi f \tag{10.7a}$$

$$\beta = \frac{\omega}{u} \tag{10.7b}$$

and

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{10.7c}$$

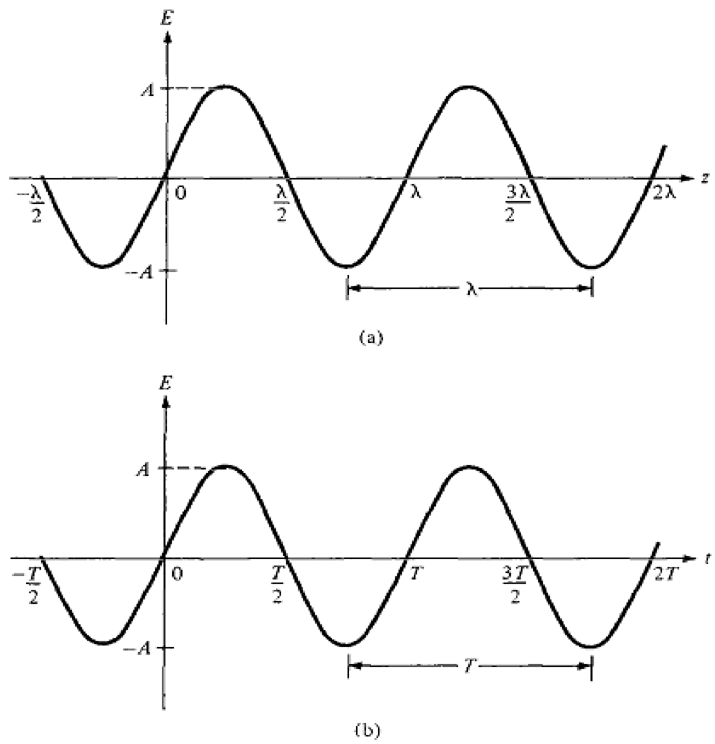


Figure 10.1 Plot of $E(z, t) = A \sin(\omega t - \beta z)$: **(a)** with constant t, **(b)** with constant z.

we expect from eqs. (10.6) and (10.7) that

$$\boxed{\beta = \frac{2\pi}{\lambda}} \tag{10.8}$$

$$\omega t - \beta z = \text{constant}$$

or

$$\frac{dz}{dt} = \frac{\omega}{\beta} = u \tag{10.9}$$

TABLE 10.1 Electromagnetic Spectrum

EM Phenomena	Examples of Uses	Approximate Frequency Range
Cosmic rays	Physics, astronomy	10^{14} GHz and above
Gamma rays	Cancer therapy	10^{10}–10^{13} GHz
X-rays	X-ray examination	10^{8}–10^{9} GHz
Ultraviolet radiation	Sterilization	10^{6}–10^{8} GHz
Visible light	Human vision	10^{5}–10^{6} GHz
Infrared radiation	Photography	10^{3}–10^{4} GHz
Microwave waves	Radar, microwave relays, satellite communication	3–300 GHz
Radio waves	UHF television	470–806 MHz
	VHF television, FM radio	54–216 MHz
	Short-wave radio	3–26 MHz
	AM radio	535–1605 kHz

EXAMPLE 10.1

The electric field in free space is given by

$$\mathbf{E} = 50\cos(10^8 t + \beta x)\,\mathbf{a}_y \text{ V/m}$$

(a) Find the direction of wave propagation.

(b) Calculate β and the time it takes to travel a distance of $\lambda/2$.

(c) Sketch the wave at $t = 0$, $T/4$, and $T/2$.

Solution:

(a) From the positive sign in $(\omega t + \beta x)$, we infer that the wave is propagating along $-\mathbf{a}_x$. This will be confirmed in part (c) of this example.

(b) In free space, $u = c$.

$$\beta = \frac{\omega}{c} = \frac{10^8}{3 \times 10^8} = \frac{1}{3}$$

or

$$\beta = 0.3333 \text{ rad/m}$$

If T is the period of the wave, it takes T seconds to travel a distance λ at speed c. Hence to travel a distance of $\lambda/2$ will take

$$t_1 = \frac{T}{2} = \frac{1}{2}\frac{2\pi}{\omega} = \frac{\pi}{10^8} = 31.42 \text{ ns}$$

Alternatively, because the wave is traveling at the speed of light c,

$$\frac{\lambda}{2} = ct_1 \quad \text{or} \quad t_1 = \frac{\lambda}{2c}$$

But

$$\lambda = \frac{2\pi}{\beta} = 6\pi$$

Hence,

$$t_1 = \frac{6\pi}{2(3 \times 10^8)} = 31.42 \text{ ns}$$

as obtained before.

(c) At $t = 0, \quad E_y = 50 \cos \beta x$

At $t = T/4, E_y = 50 \cos\left(\omega \cdot \frac{2\pi}{4\omega} + \beta x\right) = 50 \cos(\beta x + \pi/2)$

$= -50 \sin \beta x$

At $t = T/2, E_y = 50 \cos\left(\omega \cdot \frac{2\pi}{2\omega} + \beta x\right) = 50 \cos(\beta x + \pi)$

$= -50 \cos \beta x$

PRACTICE EXERCISE 10.1

In free space, $\mathbf{H} = 0.1 \cos(2 \times 10^8 t - kx)\, \mathbf{a}_y$ A/m. Calculate

(a) k, λ, and T

(b) The time t_1 it takes the wave to travel $\lambda/8$

Answer: (a) 0.667 rad/m, 9.425 m, 31.42 ns, (b) 3.927 ns,

WAVE PROPAGATION IN LOSSY DIELECTRICS

A **lossy dielectric** is a medium in which an EM wave loses power as it propagates due to poor conduction.

In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free ($\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations (see Table 9.2) become

$$\nabla \cdot \mathbf{E}_s = 0 \tag{10.11}$$

$$\nabla \cdot \mathbf{H}_s = 0 \tag{10.12}$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \tag{10.13}$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\varepsilon)\mathbf{E}_s \tag{10.14}$$

PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\varepsilon$. It is a special case of that in Section 10.3 except that

$$\boxed{\sigma \simeq 0, \qquad \varepsilon = \varepsilon_o\varepsilon_r, \qquad \mu = \mu_o\mu_r} \tag{10.42}$$

Substituting these into eqs. (10.23) and (10.24) gives

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu\varepsilon} \tag{10.43a}$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.43b}$$

Also

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}\,\underline{/0^\circ} \tag{10.44}$$

and thus **E** and **H** are in time phase with each other.

Examples

PLANE WAVES IN FREE SPACE

This is a special case of what we considered in Section 10.3. In this case,

$$\boxed{\sigma = 0, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o} \tag{10.45}$$

This may also be regarded as a special case of Section 10.4. Thus we simply replace ε by ε_o and μ by μ_o in eq. (10.43) or we substitute eq. (10.45) directly into eqs. (10.23) and (10.24). Either way, we obtain

$$\alpha = 0, \qquad \beta = \omega\sqrt{\mu_o\varepsilon_o} = \frac{\omega}{c} \tag{10.46a}$$

$$u = \frac{1}{\sqrt{\mu_o\varepsilon_o}} = c, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.46b}$$

where $c \simeq 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM wave travels in free space at the speed of light is significant. It shows that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters in eq. (10.45) into eq. (10.33), $\theta_\eta = 0$ and $\eta = \eta_o$, where η_o is called the *intrinsic impedance of free space* and is given by

$$\boxed{\eta_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = 120\pi \simeq 377\ \Omega} \tag{10.47}$$

$$\mathbf{E} = E_o \cos(\omega t - \beta z)\, \mathbf{a}_x \tag{10.48a}$$

then

$$\mathbf{H} = H_o \cos(\omega t - \beta z)\, \mathbf{a}_y = \frac{E_o}{\eta_o} \cos(\omega t - \beta z)\, \mathbf{a}_y \tag{10.48b}$$

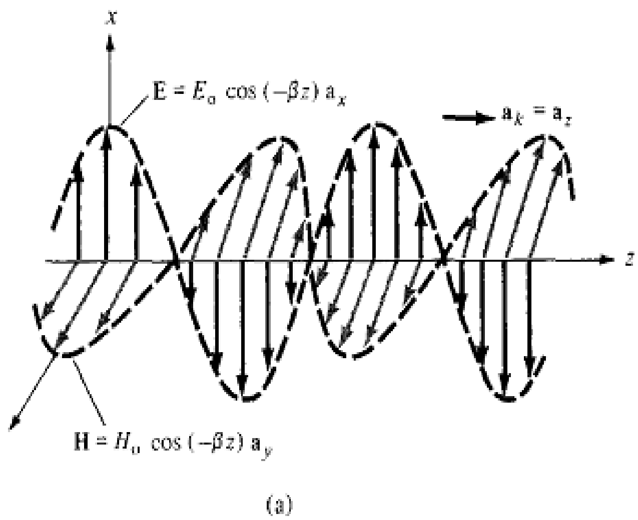
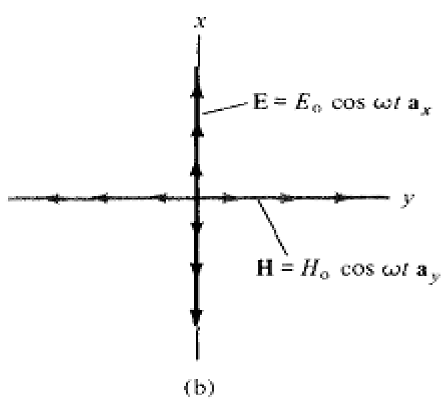


Figure 10.7 (**a**) Plot of **E** and **H** as functions of z at $t = 0$; (**b**) plot of **E** and **H** at $z = 0$. The arrows indicate instantaneous values.



$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$$

$$\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

PLANE WAVES IN GOOD CONDUCTORS

This is another special case of that considered in Section 10.3. A perfect, or good conductor, is one in which $\sigma \gg \omega\varepsilon$ so that $\sigma/\omega\varepsilon \rightarrow \infty$; that is,

$$\boxed{\sigma \simeq \infty, \qquad \varepsilon = \varepsilon_o, \qquad \mu = \mu_o\mu_r} \tag{10.50}$$

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \tag{10.51a}$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \qquad \lambda = \frac{2\pi}{\beta} \tag{10.51b}$$

Also,

$$\eta = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \tag{10.52}$$

and thus **E** leads **H** by 45°. If

$$\mathbf{E} = E_o e^{-\alpha z} \cos(\omega t - \beta z)\, \mathbf{a}_x \tag{10.53a}$$

then

$$\mathbf{H} = \frac{E_o}{\sqrt{\dfrac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ)\, \mathbf{a}_y \tag{10.53b}$$

Therefore, as **E** (or **H**) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ, shown in Figure 10.8, through which the wave amplitude decreases by a factor e^{-1} (about 37%) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_o e^{-\alpha\delta} = E_o e^{-1}$$

or

$$\boxed{\delta = \frac{1}{\alpha}} \tag{10.54a}$$

The **skin depth** is a measure of the depth to which an **EM** wave can penetrate the medium.

Equation (10.54a) is generally valid for any material medium. For good conductors, eqs. (10.51a) and (10.54a) give

$$\delta = \frac{1}{\sqrt{\pi f\mu\sigma}} \tag{10.54b}$$

EXAMPLE 11.1

Let us express $\mathcal{E}_y(z,t) = 100\cos(10^8 t - 0.5z + 30°)$ V/m as a phasor.

Solution. We first go to exponential notation,

$$\mathcal{E}_y(z,t) = \text{Re}\left[100e^{\,j(10^8 t - 0.5z + 30°)}\right]$$

and then drop Re and suppress $e^{j10^8 t}$, obtaining the phasor

$$E_{ys}(z) = 100e^{-j0.5z + j30°}$$

EXAMPLE 11.2

Given the complex amplitude of the electric field of a uniform plane wave, $\mathbf{E}_0 = 100\mathbf{a}_x + 20\angle 30°\mathbf{a}_y$ V/m, construct the phasor and real instantaneous fields if the wave is known to propagate in the forward z direction in free space and has frequency of 10 MHz.

Solution. We begin by constructing the general phasor expression:

$$\mathbf{E}_s(z) = \left[100\mathbf{a}_x + 20e^{j30°}\mathbf{a}_y\right]e^{-jk_0 z}$$

where $k_0 = \omega/c = 2\pi \times 10^7/3 \times 10^8 = 0.21$ rad/m. The real instantaneous form is then found through the rule expressed in Eq. (19):

$$\begin{aligned}\mathcal{E}(z,t) &= \text{Re}\left[100e^{-j0.21z}e^{j2\pi\times10^7 t}\mathbf{a}_x + 20e^{j30°}e^{-j0.21z}e^{j2\pi\times10^7 t}\mathbf{a}_y\right]\\ &= \text{Re}\left[100e^{j(2\pi\times10^7 t - 0.21z)}\mathbf{a}_x + 20e^{j(2\pi\times10^7 t - 0.21z + 30°)}\mathbf{a}_y\right]\\ &= 100\cos(2\pi\times10^7 t - 0.21z)\mathbf{a}_x + 20\cos(2\pi\times10^7 t - 0.21z + 30°)\,\mathbf{a}_y\end{aligned}$$

WAVE PROPAGATION IN DIELECTRICS

We now extend our analytical treatment of the uniform plane wave to propagation in a dielectric of permittivity ϵ and permeability μ. The medium is assumed to be homogeneous (having constant μ and ϵ with position) and isotropic (in which μ and ϵ are invariant with field orientation). The Helmholtz equation is

$$\nabla^2\mathbf{E}_s = -k^2\mathbf{E}_s \tag{36}$$

where the wavenumber is a function of the material properties, as described by μ and ϵ:

$$k = \omega\sqrt{\mu\epsilon} = k_0\sqrt{\mu_r\epsilon_r} \tag{37}$$

For E_{xs} we have

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs} \tag{38}$$

An important feature of wave propagation in a dielectric is that k can be complex-valued, and as such it is referred to as the complex *propagation constant*. A general solution of (38), in fact, allows the possibility of a complex k, and it is customary to write it in terms of its real and imaginary parts in the following way:

$$jk = \alpha + j\beta \tag{39}$$

A solution to (38) will be:

$$E_{xs} = E_{x0} e^{-jkz} = E_{x0} e^{-\alpha z} e^{-j\beta z} \tag{40}$$

Multiplying (40) by $e^{j\omega t}$ and taking the real part yields a form of the field that can be more easily visualized:

$$E_x = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z) \tag{41}$$

The ways in which physical processes in a material can affect the wave electric field are described through a *complex permittivity* of the form

$$\epsilon = \epsilon' - j\epsilon'' = \epsilon_0(\epsilon'_r - j\epsilon''_r) \tag{42}$$

We can substitute (42) into (37), which results in

$$k = \omega\sqrt{\mu(\epsilon' - j\epsilon'')} = \omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\epsilon''}{\epsilon'}} \tag{43}$$

Note the presence of the second radical factor in (43), which becomes unity (and real) as ϵ'' vanishes. With nonzero ϵ'', k is complex, and so losses occur which are quantified through the attenuation coefficient, α, in (39). The phase constant, β (and consequently the wavelength and phase velocity), will also be affected by ϵ''. α and β are found by taking the real and imaginary parts of jk from (43). We obtain:

$$\alpha = \mathrm{Re}\{jk\} = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1\right)^{1/2} \tag{44}$$

$$\beta = \mathrm{Im}\{jk\} = \omega\sqrt{\frac{\mu\epsilon'}{2}}\left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1\right)^{1/2} \tag{45}$$

Whether or not losses occur, we see from (41) that the wave phase velocity is given by

$$v_p = \frac{\omega}{\beta} \tag{46}$$

The wavelength is the distance required to effect a phase change of 2π radians

$$\beta\lambda = 2\pi$$

which leads to the fundamental definition of wavelength,

$$\lambda = \frac{2\pi}{\beta} \tag{47}$$

Because we have a uniform plane wave, the magnetic field is found through

$$H_{ys} = \frac{E_{x0}}{\eta} e^{-\alpha z} e^{-j\beta z}$$

where the intrinsic impedance is now a complex quantity,

$$\eta = \sqrt{\frac{\mu}{\epsilon' - j\epsilon''}} = \sqrt{\frac{\mu}{\epsilon'}} \frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} \tag{48}$$

The electric and magnetic fields are no longer in phase.

A special case is that of a lossless medium, or *perfect dielectric,* in which $\epsilon'' = 0$, and so $\epsilon = \epsilon'$. From (44), this leads to $\alpha = 0$, and from (45),

$$\beta = \omega\sqrt{\mu\epsilon'} \qquad \text{(lossless medium)} \tag{49}$$

With $\alpha = 0$, the real field assumes the form

$$E_x = E_{x0}\cos(\omega t - \beta z) \tag{50}$$

We may interpret this as a wave traveling in the $+z$ direction at a phase velocity v_p, where

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon'}} = \frac{c}{\sqrt{\mu_r\epsilon'_r}}$$

The wavelength is

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\epsilon'}} = \frac{1}{f\sqrt{\mu\epsilon'}} = \frac{c}{f\sqrt{\mu_r\epsilon'_r}} = \frac{\lambda_0}{\sqrt{\mu_r\epsilon'_r}} \quad \text{(lossless medium)} \tag{51}$$

where λ_0 is the free space wavelength. Note that $\mu_r\epsilon'_r > 1$, and therefore the wavelength is shorter and the velocity is lower in all real media than they are in free space.

Associated with E_x is the magnetic field intensity

$$H_y = \frac{E_{x0}}{\eta}\cos(\omega t - \beta z)$$

where the intrinsic impedance is

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{52}$$

D11.3. A 9.375-GHz uniform plane wave is propagating in polyethylene (see Appendix C). If the amplitude of the electric field intensity is 500 V/m and the material is assumed to be lossless, find: (a) the phase constant; (b) the wavelength in the polyethylene; (c) the velocity of propagation; (d) the intrinsic impedance; (e) the amplitude of the magnetic field intensity.

Ans. 295 rad/m; 2.13 cm; 1.99×10^8 m/s; 251 Ω; 1.99 A/m

EXAMPLE 11.3

Let us apply these results to a 1-MHz plane wave propagating in fresh water. At this frequency, losses in water are negligible, which means that we can assume that $\epsilon'' \doteq 0$. In water, $\mu_r = 1$ and at 1 MHz, $\epsilon'_r = 81$.

Solution. We begin by calculating the phase constant. Using (45) with $\epsilon'' = 0$, we have

$$\beta = \omega\sqrt{\mu\epsilon'} = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\epsilon'_r} = \frac{\omega\sqrt{\epsilon'_r}}{c} = \frac{2\pi \times 10^6\sqrt{81}}{3.0 \times 10^8} = 0.19 \text{ rad/m}$$

Using this result, we can determine the wavelength and phase velocity:

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{.19} = 33 \text{ m}$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{.19} = 3.3 \times 10^7 \text{ m/s}$$

The wavelength in air would have been 300 m. Continuing our calculations, we find the intrinsic impedance using (48) with $\epsilon'' = 0$:

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} = \frac{\eta_0}{\sqrt{\epsilon'_r}} = \frac{377}{9} = 42\ \Omega$$

If we let the electric field intensity have a maximum amplitude of 0.1 V/m, then

$$E_x = 0.1\cos(2\pi 10^6 t - .19z) \text{ V/m}$$

$$H_y = \frac{E_x}{\eta} = (2.4 \times 10^{-3})\cos(2\pi 10^6 t - .19z) \text{ A/m}$$

10.7 Sea water plays a vital role in the study of submarine communications. Assuming that for sea water, $\sigma = 4$ S/m, $\varepsilon_r = 80$, $\mu_r = 1$, and $f = 100$ MHz, calculate: (a) the phase velocity, (b) the wavelength, (c) the skin depth, (d) the intrinsic impedance.

10.8 In a certain medium with $\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$,

$$\mathbf{H} = 12e^{-0.1y} \sin (\pi \times 10^8 t - \beta y)\, \mathbf{a}_x \text{ A/m}$$

find: (a) the wave period T, (b) the wavelength λ, (c) the electric field $\mathbf{E}$, (d) the phase difference between $\mathbf{E}$ and $\mathbf{H}$.

10.9 In a medium,

$$\mathbf{E} = 16e^{-0.05x} \sin (2 \times 10^8 t - 2x)\, \mathbf{a}_z \text{ V/m}$$

find: (a) the propagation constant, (b) the wavelength, (c) the speed of the wave, (d) the skin depth.

10.10 A uniform wave in air has

$$\mathbf{E} = 10 \cos (2\pi \times 10^6 t - \beta z)\, \mathbf{a}_y$$

(a) Calculate β and λ.

(b) Sketch the wave at $z = 0, \lambda/4$.

(c) Find $\mathbf{H}$.

10.11 The magnetic field component of an EM wave propagating through a nonmagnetic medium ($\mu = \mu_o$) is

$$\mathbf{H} = 25 \sin (2 \times 10^8 t + 6x)\, \mathbf{a}_y \text{ mA/m}$$

Determine:

(a) The direction of wave propagation.

(b) The permittivity of the medium.

(c) The electric field intensity.

10.12 If $\mathbf{H} = 10 \sin (\omega t - 4z)\mathbf{a}_x$ mA/m in a material for which $\sigma = 0$, $\mu = \mu_o$, $\varepsilon = 4\varepsilon_o$, calculate ω, λ, and $\mathbf{J}_d$.

Prob. 10.7

$$\frac{\sigma}{\omega\varepsilon} = \frac{4}{2\pi x10^5 x81x10^{-9}/36\pi} = \frac{80,000}{9} >> 1$$

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{2\pi x10^5}{2} x4\pi x10^{-7} x4} = 0.4\pi$$

(a) $u = \omega/\beta = \dfrac{2\pi x10^5}{0.4\pi} = \underline{\underline{5x10^5}}$ m/s

(b) $\lambda = 2\pi/\beta = \dfrac{2\pi}{0.4\pi} = \underline{\underline{5}}$ m

(c) $\delta = 1/\alpha = \dfrac{1}{0.4\pi} = \underline{\underline{0.796}}$ m

(d) $\eta = |\eta|\angle\theta_\eta, \theta_\eta = 45^o$

$$|\eta| = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt[4]{1+\left(\frac{\sigma}{\omega\varepsilon}\right)^2}} \cong \sqrt{\frac{\mu}{\varepsilon}\frac{\omega\varepsilon}{\sigma}} = \sqrt{\frac{4\pi x10^{-7} x2\pi x10^8}{4}} = 14.05$$

$\eta = \underline{\underline{14.05\angle 45^o \quad \Omega}}$

Prob. 10.8 (a)

$$T = 1/f = 2\pi/\omega = \frac{2\pi}{\pi x 10^8} = \underline{\underline{20 \text{ ns}}}$$

(b) Let $x = \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2}$

$$\frac{\alpha}{\beta} = \left(\frac{x-1}{x+1}\right)^{1/2}$$

But $\alpha = \frac{\omega}{c}\sqrt{\frac{\mu_r\varepsilon_r}{2}}\sqrt{x-1}$

$$\sqrt{x-1} = \frac{\alpha c}{\omega\sqrt{\frac{\mu_r\varepsilon_r}{2}}} = \frac{0.1x3x10^8}{\pi x 10^8\sqrt{2}} = 0.06752 \longrightarrow x = 1.0046$$

$$\beta = \left(\frac{x+1}{x-1}\right)^{1/2}\alpha = \left(\frac{2.0046}{0.0046}\right)^{1/2} 0.1 = 2.088$$

$$\lambda = 2\pi/\beta = \frac{2\pi}{2.088} = \underline{\underline{3}}\text{ m}$$

(c) $|\eta| = \frac{\sqrt{\mu/\varepsilon}}{\sqrt{x}} = \frac{377}{2\sqrt{1.0046}} = 188.1$

$$x = \sqrt{1 + \left(\frac{\sigma}{\omega\varepsilon}\right)^2} = 1.0046$$

$$\frac{\sigma}{\omega\varepsilon} = 0.096 = \tan 2\theta_\eta \longrightarrow \theta_\eta = 2.74^o$$

$$\eta = 188.1\angle 2.74^o \quad \Omega$$

Prob. 10.9 (a) $\gamma = \alpha + j\beta = \underline{\underline{0.05 + j2}}$ /m

(b) $\lambda = 2\pi/\beta = \pi = \underline{\underline{3.142}}$ m

(c) $u = \omega/\beta = \dfrac{2x10}{2} = \underline{\underline{10^8}}$ m/s

(d) $\delta = 1/\alpha = \dfrac{1}{0.05} = \underline{\underline{20}}$ m

Prob. 10.10 (a) $\beta = \omega/c = \dfrac{2\pi x 10^6}{3x10^8} = \underline{\underline{0.02094}}$ rad/m,

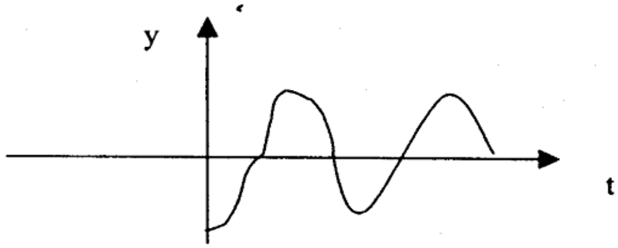
$\lambda = 2\pi/\beta = \underline{\underline{300}}$ m

(b) When $z = 0, \quad E_y = 10\cos\omega t$

$z = \lambda/4, \quad E_y = 10\cos(\omega t - \frac{2\pi}{\lambda}\frac{\lambda}{4}) = 10\sin\omega t$

$z = \lambda/2, \quad E_y = 10\cos(\omega t - \pi) = -10\cos\omega t$

$z = \lambda/2$



(c)

$$H = \frac{1}{120\pi}\cos(2\pi x 10^6 t - 2\pi z/300)a_x = \underline{\underline{26.53\cos(2\pi x 10^6 t - 0.02094)a_x}} \text{ A/m}$$

Prob. 10.11 (a) Along -x direction.

(b) $\beta = 6, \quad \omega = 2x10^{8}$.

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r}$$

$$\sqrt{\varepsilon_r} = \beta c/\omega = \frac{6x3x10^{8}}{2x10^{8}} = 9 \longrightarrow \varepsilon_r = 81$$

$$\varepsilon = \varepsilon_o\varepsilon_r = \frac{10^{-9}}{36\pi}x81 = \underline{\underline{7.162x10^{-10}}} \text{ F/m}$$

(c) $\eta = \sqrt{\mu/\varepsilon} = \sqrt{\mu_o/\varepsilon_o}\sqrt{\mu_r/\varepsilon_r} = \frac{120\pi}{9}$

$$E_o = H_o\eta = 25x10^{-3}x377/9 = 1.047$$

$$a_E x a_H = a_k \longrightarrow a_E x a_y = -a_x \longrightarrow a_E = a_z$$

$$E = \underline{\underline{1.047\sin(2x10^{8}t + 6x)a_z}} \text{ V/m}$$

Prob. 10.12 $\beta = 4 \longrightarrow \lambda = 2\pi / \beta = \underline{\underline{1.571}}$ m

Also, $\beta = \omega / u = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{\mu_r \varepsilon_r}$

$$\omega = \frac{\beta c}{\sqrt{\mu_r \varepsilon_r}} = \frac{4x3x10^8}{\sqrt{4}} = \underline{\underline{6x10^8}} \text{ rad/s}$$

$$J_d = \nabla x H = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x(z) & 0 & 0 \end{vmatrix} = \frac{\partial H_x}{\partial z} a_y$$

$$J_d = -40\cos(\omega t - 4z)x10^{-3} a_y = \underline{\underline{-40\cos(\omega t - 4z) a_y}} \text{ mA/m}^2$$

Electromagnetic Spectrum

What is the Electromagnetic Spectrum?

Scientists have found that many types of wave can be arranged together like notes on a piano keyboard, to form a scale

How The Waves Fit Into The Spectrum

You need to remember:

the names of the waves,

which ones have high or low frequencies

the order they fit into the electromagnetic spectrum

All of these are "electromagnetic waves". This means that although they appear to be very different, in fact they're all made of the same kind of vibrations - but at different .

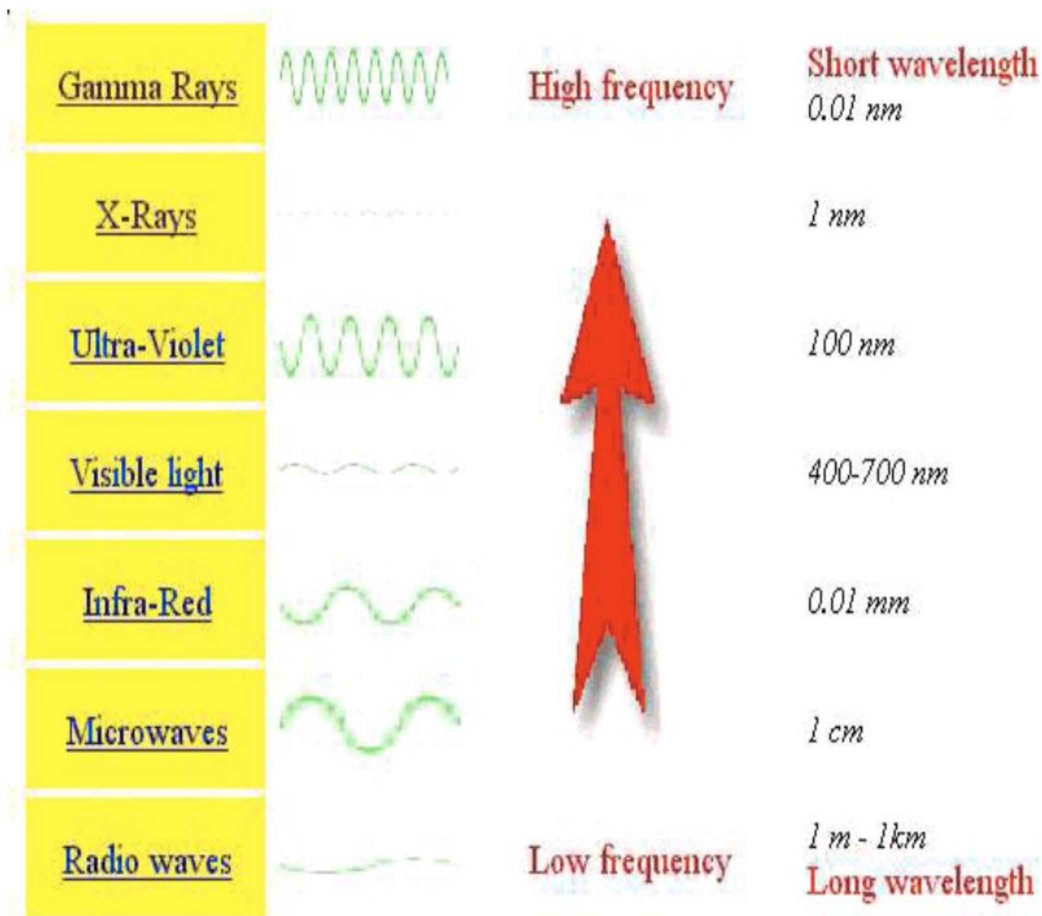
All electromagnetic waves travel at the same speed - 300,000,000 metres per second, which is the speed of light.

What does "frequency" mean?

The frequency of a wave is the number of waves per second.

What does "wavelength" mean?

The wavelength is the distance from the peak of one wave to the next one.



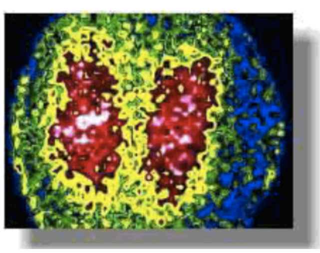
EM- Spectrum Applications

Gamma Rays

How they are made: Gamma rays are given off by stars, and by some radioactive substances. They are extremely high frequency waves, and carry a large amount of energy. They pass through most materials, and are quite difficult to stop - you need lead or concrete in order to block them out.

Uses: Because Gamma rays can kill living cells, they are used to kill cancer cells without having to resort to difficult surgery. This is called "Radiotherapy", and works because cancer cells can't repair themselves like healthy cells can when damaged by gamma rays. Getting the dose right is very important!

Doctors can put slightly radioactive substances into a patient's body, then scan them to detect the gamma rays and build up a picture of what's going on inside the patient. These are called "tracers". This is very useful because they can see the body processes actually working, rather than just looking at still pictures.



The picture on the left is a "Scintigram", and shows an asthmatic person's lungs. The patient was given a slightly radioactive gas to breathe, and the picture was taken using a gamma camera to detect the radiation. The colours show the air flow in the lungs.

In industry, radioactive "tracer" substances can be put into pipes and machinery, then we can detect where the substances go. This is basically the same use as in medicine.

Gamma rays kill microbes, and are used to sterilize food so that it will keep fresh for longer. This is known as "irradiated" food.

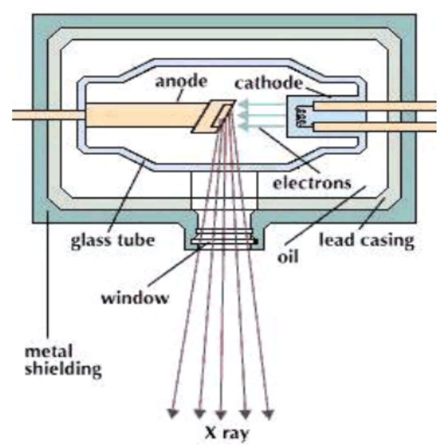
Gamma rays are also used to sterilize medical equipment.

Dangers: Gamma rays cause cell damage and can cause a variety of cancers. They cause mutations in growing tissues, so unborn babies are especially vulnerable.

Contents

X-Rays

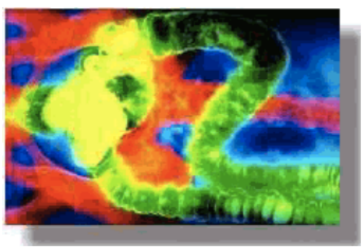
How they are made: X-rays are very high frequency waves, and carry a lot of energy. They will pass through most substances, and this makes them useful in medicine and industry to see inside things.



X-rays are given off by stars, and strongly by some types of nebula.
When we use X-rays, we make them by firing a beam of electrons at a "target". If we fire the electrons with enough energy, X-rays will be produced.

Uses: X-rays are used by doctors to see inside people. They pass easily through soft tissues, but not so easily through bones. We send a beam of X-Rays through the patient and onto a piece of film, which goes dark where X-Rays hit it. This leaves white patches on the film where the bones were in the way.

Lower energy X-Rays don't pass through tissues as easily, and can be used to scan soft areas such as the brain



Sometimes a doctor will give a patient a "Barium Meal", which is a drink of Barium Sulphate. This will absorb X-rays, and so the patient's intestines will show up clearly on a X-Ray image.

X-Rays are also used in airport security checks, to see inside your luggage. They are also used by astronomers - many objects in the universe emit X-rays, which we can detect using suitable radio telescopes.

Dangers: X-Rays can cause cell damage and cancers. This is why Radiographers in hospitals stand behind a shield when they X-ray their patients. Although the dose is not enough to put the patient at risk, they take many images each day and could quickly build up a dangerous dose themselves.

Ultra-Violet

How they are made: Ultra-Violet light is made by special lamps, for example, on sun beds. It is also given off by the Sun in large quantities.
We call it "UV" for short.

Uses for UV light include getting a sun tan, detecting forged bank notes in shops, and hardening some types of dental filling.
You also see UV lamps in discos, where they make your clothes glow. This happens because substances in washing powder "fluoresce" when UV light strikes them.
When you mark your posessions with a security marker pen, the ink is invisible unless you shine a UV lamp at it.

Ultraviolet rays can be used to kill microbes. Hospitals use UV lamps to sterilise surgical equipment and the air in operating theatres. Food and drug companies also use UV lamps to sterilise their products

Suitable doses of Ultraviolet rays cause the body to produce vitamin D, and this is used by doctors to treat vitamin D deficiency and some skin disorders.

Dangers: Large doses of UV can damage the retinas in your eyes, so it's important to check that your sunglasses will block UV light. The cheaper sunglasses tend not to protect you against UV, and this can be really dangerous. When you wear sunglasses the pupils of your eye get bigger, because less light reaches them. This means that if your sunglasses don't block UV, you'll actually get more ultra-violet light on your retinas than if you didn't wear them.

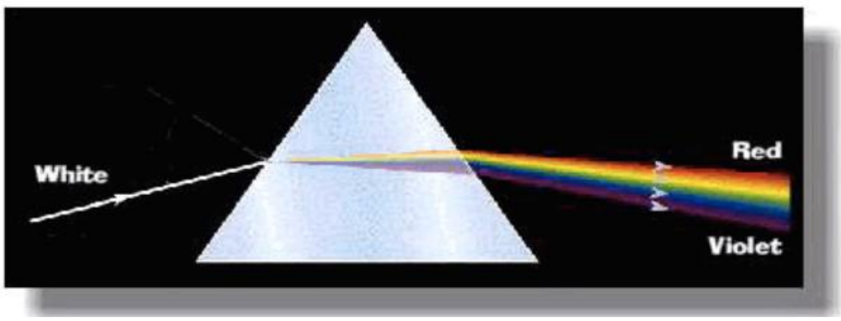
Large doses of UV cause sunburn and even skin cancer. Fortunately, the ozone layer in the Earth's atmosphere screens us from most of the UV given off by the Sun. Think of a sun tan as a radiation burn!

Visible Light

How it is made: Our eyes can detect only a tiny part of the electromagnetic spectrum, called visible light. This means that there's a great deal happening around us that we're simply not aware of, unless we have instruments to detect it.

Light waves are given off by anything that's hot enough to glow. This is how light bulbs work - an electric current heats the lamp filament to around 3,000 degrees, and it glows white-hot. The surface of the Sun is around 5,600 degrees, and it gives off a great deal of light.

White light is actually made up of a whole range of colors, mixed together.
We can see this if we pass white light through a glass prism - the violet light is bent ("refracted") more than the red, because it has a shorter wavelength - and we see a rainbow of colors.



Uses: We use light to see things! As the Sun sends so much light towards our planet, we've evolved to make use of those particular wavelengths in order to sense our environment.

Light waves can also be made using a laser. This works differently to a light bulb, and produces "coherent" light. Lasers are used in Compact Disc players, where the light is reflected from the tiny pits in the disc, and the pattern is detected and translated into sound or data.

Lasers are also used in laser printers, and in aircraft weapon aiming systems.

Dangers: Too much light can damage the retina in your eye. This can happen when you look at something very bright, such as the Sun. Although the damage can heal, if it's is too bad it'll be permanent.

How they are made: Infra-red waves are just below visible red light in the electromagnetic spectrum ("Infra" means "below"). You probably think of Infra-red waves as heat, because they're given off by hot objects, and you can feel them as warmth on your skin. Infra-Red waves are also given off by stars, lamps, flames and anything else that's warm - including you.

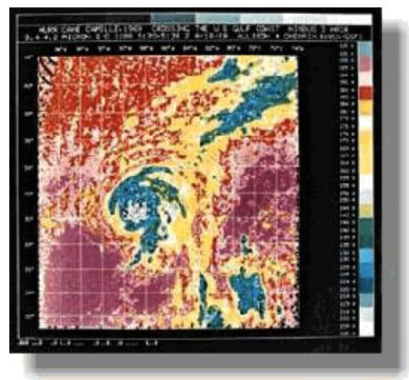
Uses:

Infra-red waves are called "IR" for short. They are used for many tasks, for example, remote controls for TVs and video recorders, and physiotherapists use heat lamps to help heal sports injuries.

Because every object gives off IR waves, we can use them to "see in the dark". Night sights for weapons sometimes use a sensitive IR detector (other types, called "image intensifiers", use visible light). Remember the film, "Predator"?

Apart from remote controls, one of the most common modern uses for IR is in the field of security. "Passive Infra-Red" (PIR) detectors are used in burglar alarm systems, and to control the security lighting that many people have fitted outside their houses. These detect the Infra-Red emitted by people and animals. You've probably seen TV programmes in which police helicopters track criminals at night, using cameras which can see in the dark. These cameras use Infra-Red waves instead of "ordinary" light, which is why people look bright in these pictures.

Weather forecasters use satellite pictures to see what's heading our way. Some of the images they use are taken using IR cameras, because they show cloud and rain patterns more clearly.



Dangers:

The danger from too much Infra-Red radiation is very simple - it makes you hot.

Microwaves

How they are made: Microwaves are basically extremely high frequency radio waves, and are made by various types of transmitter. In a mobile phone, they're made by a transmitter chip and an antenna, in a microwave oven they're made by a "magnetron". Their wavelength is usually a couple of centimetres. Stars also give off microwaves.

Uses: Microwaves cause water and fat molecules to vibrate, which makes the substances hot. Thus we can use microwaves to cook many types of food.

Mobile phones use microwaves, as they can be generated by a small antenna, which means that the phone doesn't need to be very big. The drawback is that, being small, they can't put out much power, and they also need a line of sight to the transmitter. This means that mobile phone companies need to have many transmitter towers if they're going to attract customers.

Microwaves are also used by traffic speed cameras, and for radar, which is used by aircraft, ships and weather forcasters.

The most common type of radar works by sending out bursts of microwaves, detecting the "echoes" coming back from the objects they hit, and using the time it takes for the echoes to come back to work out how far away the object is.

Dangers: Prolonged exposure to microwaves is known to cause "cataracts" in your eyes, which is a clouding of the cornea. So don't make a habit of pressing your face against the microwave oven door to see if your food's ready!

Recent research indicates that microwaves from mobile phones can affect parts of your brain - after all, you're holding the transmitter right by your head. The current advice is to keep calls short.

People who work on aircraft carrier decks wear special suits which reflect microwaves, to avoid being "cooked" by the powerful radar units in modern military planes.

Radio Waves

How they are made:

Radio waves are made by various types of transmitter, depending on the wavelength. They are also given off by stars, sparks and lightning, which is why you hear interference on your radio in a thunderstorm.

Uses:

Radio waves are the lowest frequencies in the electromagnetic spectrum, and are used mainly for communications.

They are divided into:-

- Long Wave, around 1~2 km in wavelength. The radio station "Atlantic 252" broadcasts here.
- Medium Wave, around 100m in wavelength, used by BBC Radio 5 and other "AM" stations.

- VHF, which stands for "Very High Frequency" and has wavelengths of around 2m. This is where you find stereo "FM" radio stations, such as "Galaxy 101" and "GWR FM". Further up the VHF band are civilian aircraft and taxis.
- UHF stands for "Ultra High Frequency", and has wavelengths of less than a metre. It's used for Police radio communications, military aircraft radios and television transmissions.

Dangers:

Large doses of radio waves are believed to cause cancer, leukaemia and other disorders. Some people claim that the very low frequency field from overhead power cables near their homes has affected their health.